Geometrical Logic

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G E O M E T R I C A L

L O G I C

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II

C O N T E N T S

PREFACE

Algebraic-geometrical logic or,simply geometrical logic likewise known as topologic,appeared as a spatial representation of algebraic logic (the first,preliminary outline having been published in the *Przegląd Filozoficzny* (Philosophical Review),Warsaw 1926 and 1927). The representation was effected by the application of Descartes's co-ordinates to logic and by making use of the correspondences between duality in logic and in (projective) geometry. The spatialization of logic enabled us to give it a marked structural and architectonic character - one which brought out the "order" which reigns in this domain. This geometrical logic is of great philosophical,and particularly of ontological,significance; the whole philosophical aspect has,however, been disregarded in the present work. Only the foundations of geometrical logic as such have been dealt with,and the architectonics ruling its elements have been brought out (*inter alia*,the concept of harmonic sets of four and that of the neutral mean,etc. have been introduced).

Logic (pan-logic) can in its specifications acquire a character which is ~~distinctly~~ distinctly dynamic (dialectic) and parallel with physical geometry (non-Euclidean). In the Appendix,the correspondence between the logical sum,logical product,the logical neutral mean and the harmonic,arithmetical and geometrical means has been introduced, thus leading to the discovery of a number of hitherto unknown elementary arithmetical theorems which bind together these three Pythagorean means.

INTRODUCTION

The Idea of Geometrical Logic

Geometrical logic, the most general principles of which we shall develop below, is not isolated in the domain of science in general, and in particular in that of mathematics. For it is in mathematics above all that the age-old thought, which has with irresistible, instinctive force imposed itself on the spirit of humanity, the thought that the worlds of spatial and non-spatial elements are connected in some profound manner, has met with unqualified success.

It was in this direction that Greek arithmetic, as developed in the school of Pythagoras, first went; we see here how fruitful were the scientific results of the conception of giving spatial models of so completely non-spatial entities such as are numbers. Pythagorean arithmetic was in its entirety a geometrical arithmetic, the philosophic foundation of which was based on the postulate connecting the world of space with that of numbers, and stating that a point in space is none other than a unit (a component of all numbers) possessing a position in space. Accordingly, the Pythagoreans represented numbers as separate points, being their component units, and, according to the shapes produced by these points they distinguished triangular, square, and right-angled numbers. As a result of the spatial interpretation of whole numbers they found that by adding to unity the succeeding odd number, 3, a square could be obtained and that adding similarly to the figure and number so obtained the following odd numbers (5,7,etc.), the resulting figures and numbers were also quadratic (9,16,etc.). In this way we find at the inception of Greek arithmetic that extremely interesting theorem of the theory of numbers which says that the sum of series of consecutive odd numbers, beginning with unity will always represent a quadratic number $[1 + 3 + 5 + \ldots\ldots + (2n - 1) = n^2]$. This example shows how important a rôle the method of spatial representation of numbers played in Greek arithmetic.

This interconnection of the worlds of numbers and of space attains its culmination in the analytical geometry of Descartes, in which the multiplicity of numerical pairs (x,y) is represented as a plane geometrical figure, and of triple numbers (x,y,z) by a three-dimensional figure. The harmony of these two worlds - the analytical and the spatial - is

complete; although one of these **domains** consists of **completely** non-spatial elements,such as are numbers,and the other of elements existing in space,yet the laws and relations prevailing in these worlds are identical.

It now becomes apparent that the concept of geometrical logic is not so strange as it might at first sight have appeared. On the contrary,it is perfectly natural,and its realization is a necessary complementation of such branches of mathematics as geometrical arithmetic,geometrical algebra,and analytical geometry,for in all these cases we have to deal with the same fundamental matter. This is the correspondence of the domains of spatial and non-spatial worlds,and of the coincidence of these two **realms**,with this difference only that the non-spatial **domain** is in one case represented by numbers,and **in** the other by concepts. Actually numbers and concepts are of the same nature,and have themselves nothing in common with extension; they are par excellence of a non-intuitive,non-figurative,purely conceptional nature. If then numbers are, in spite of their non-spatial nature in such astounding,yet undoubted harmony with spatial objects,and if they can be represented in their forms,it is but a short step to the presumption that this will apply to concepts in general,and that,in the same way as geometrical arithmetic and algebra,and analytical geometry already exist,so also must exist geometrical logic. Moreover,since we already recognise the ordinary quantitative geometrical algebra,what can be more natural than to suppose that for qualitative algebra,which is identical with exact logic, there would be a corresponding certain qualitative geometry - the geometry of logic.

This conception of the affinity of the domains of thought and of space existed,although only in a rudimentary form,in the mind of the founder of logic,Plato,who first discovered the world of concepts,of logic. This world is,according to the profound convictions of its discoverer,a model of order,a cosmos,a system,in which each concept is in some exactly defined relation to any other concept,and occupies an exactly defined "position" in this system. If this is so,however,the concept of this world of logic should present an analogy to the spatial world; the elements of this logical world should be distributed and correlated in some "logical space". Thus thought Plato,when he spoke of "the space of thought" (τόπος νοητός),in which ideas were to exist.

The tendency to geometrize the logical world is undoubted in Plato; the detailed realization of this tendency was not possible at the time, since the world of ideas discovered by him was itself too little explo-

red - and it was not until two thousand years later that the genius of Leibnitz found means of furthering the exact,mathematical knowledge of this world. Leibnitz similarly attached great importance to the visualization of the elements and principles of logic by their spatial representation,by "drawing lines" (per linearum ductum),and there can be no doubt that if he had succeeded in creating a coherent system of algebraic logic he would at the same time have given us a system of the geometry of logic.

We can also find in the usual,traditional non-mathematical logic elements testifying to our instinctive,unconscious conviction as to the affinity of the worlds of thought and of space. We have here in mind the employment of diagrams for the expression of logical relationship. In general,even logicians,not to speak of those learning logic,do not realize the exceptional importance of the circumstance that the relations between non-spatial concepts can be illustrated by the relations between spatial elements,such as,for example,circles. The use of diagrams is usually regarded as an ingenuous didactic device,facilitating orientation in the relations prevailing in the realm of thought,and making it more easy for us to understand these relations than would be possible with direct consideration of the problems in question. Those holding such views would assert that the use of such diagrams is in no way strange,since the human mind actually does possess a tendency towards the spatial representation of thoughts or of spiritual entities in general. It might be replied that,in the given case,it is immaterial that such spatial representation is helpful in the teaching of logic, and that it is not the fact that we quite naturally tend to apply this means of representation,under the urge of some natural intellectual instinct,but rather we should note that the use of such diagrams withstands objective tests,and that ordinary spatial diagrams are,if even to only a limited extent,able accurately to represent thought relations. It is in this that consists the objective aspect of the above-discussed spatial correspondence of logical elements; this is a fact of the highest scientific and philosophical importance,although at first sight it appears to be so insignificant,and apparently of purely didactical-psychologic value. Yet these spatial schemes of classical logic are incontrovertible evidence of the coincidence of these two so dissimilar domains,and allow of the hope that,with the moment when logic attains the dignity of an exact science,this as yet fragmentary,disconnected,and limited representation of the world of logic in space will also be enabled to attain a fundamentally higher level. We may already,

by the appropriate selection of spatial elements (that is,above all, straight lines,and not things of a higher type,such as are circles), a priori hope to attain an adequate and systematic representation in space of the elements,operations,and relations of the world of logic, and this would be equivalent to the spatialization of exact logic,in other words with the creation of the fundamentals of exact geometrical logic.

But if we attempt to find in the ordinary,traditional logic elements of affinity of the domain of concepts with that of space,we must also discuss the language,the terminology of logic,in which our intuition of this affinity has found its unconscious expression. And so the "terms" of judgement betray their spatial origin,since they designate the limits of judgement,and in this way suggest the idea of a judgement as a segment joining two limiting points: the terms of judgement,subject and predicate. We see the same in the terminology of syllogisms,in which we speak of the "middle term",i.e.,of the term situated between two extreme ones,as if we were speaking not of concepts,but of points on a line. Expressions such as that one idea "is contained" in another,or the "crossing" of ideas - these are all redolent of the spatial factor. Further,we speak of the "definition" of concepts,i.e.,of the assignment to them of limits,of their situation in space,as it were. If it be objected that the transference of terms from the domain of space to that of logic is perfectly natural,and due to the circumstance that our knowledge of spatial objects is of more remote origin and is far more complete than is that of abstract ideas,and that it for this reason that we characterize a concept by its situation,we must of course agree to this. Yet this in no way detracts from the significant fact that these spatial terms very successfully and exactly characterize the correlation of logical elements,and that they give rise to spatial representations which correctly depict logical structures.

We thus see that indications for the geometrization of the domain of logic abound on all sides. In the domain of ordinary logic the mere fact of its terminology,so deeply imbued with spatial elements,should lead us to search not only for the subjective sources of this so significant fact,but also for its objective basis,to the existence of which testify the geometrical diagrams of this traditional logic. The possibility and necessity of the spatialization of logic are suggested by still other evidence,viz.,the relationship between the objects of logic and of mathematical analysis. Arithmetic,algebra,and analysis in general apply,similarly to logic,to a world of non-spatial elements,and yet geo-

metrical arithmetic,geometrical algebra,and geometrical analysis (known as analytical geometry) exist - why therefore should geometrical logic not be possible? This is the more probable as the mere fact of the existence of logic in the form of qualitative algebra (algebra of logic) requires the creation of a geometrical counterpart of this qualitative algebra,such as we already possess,to a certain extent,for ordinary algebra. And such an analogue would be termed geometrical logic.

CHAPTER I.

Geometrization of the Axioms of Algebraic Logic

Before we undertake an examination of the features proper of geometrical logic (also called by us topologic), we must first deal more closely with the algebraic logic which we are to geometrize. Algebraic logic (the algebra of logic) is a system based on a number of axioms which permit us to deduce further theorems. We shall base ourselves in the following representation of the algebra of logic on the first system of axioms, given by E.Huntington in his Sets of Independent Postulates for the Algebra of Logic[1].

Huntington takes into consideration a multiplicity of elements which contain at least two elements differing from each other; moreover assuming that the elements a and b belong to this system, the elements $a + b$ and ab (the logical sum and product of a and b) will likewise belong to it. (Huntington formulates the above properties of the system of elements by means of two postulates.)

The remaining postulates which govern this system of elements can be expressed as follows:

1^a. There is an element 0, such that $a + 0 = a$ for every element a.

1^b. There is an element 1, such that $a.1 = a$ for every element a.

2^a. $a + b = b + a$

2^b. $ab = ba$

3^a. $a + bc = (a + b)(a + c)$

3^b. $a(b + c) = ab + ac$

4. There is an element a' such that for every element a:

4^a. $a + a' = 1$, and

4^b. $aa' = 0$

Postulate No. 1. The elements 0 and 1 are two limitary elements of the system of algebraic logic as regards which we shall later ascertain that 0 is the smallest logical comprehension and 1 the largest (cf. p. 32, theorem 12). Postulate No. 1 states the existence of these elements as the moduli of addition and multiplication, i.e. such elements (as 0 and 1 in arithmetic) which do not change the elements to which

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[1] Transactions of the American Mathematical Society, Vol. 5, 1904, pp. 292 - 296.

they are joined by the symbol "+" (modulus 0) or by the symbol "x" (modulus 1).

The essential nature of logical moduli can be better understood if we bear in mind that 0 is the smallest and 1 the largest logical comprehension. The smallest, poorest, least determinate, logical comprehension is the concept: "object" ("being" = "something"). If so, to say a given element is a, is clearly equivalent to stating that it is a + 0, since a + 0 signifies "object a". The formula a.1 = a, in line with the significance of logical multiplication expresses that when we seek the greatest part common for the logical comprehension of a and 1, we find it equals a. As 1 is the largest comprehension, the richest, the one which involves in itself all the other comprehensions, the comprehension a is naturally the greatest part common to a and 1.

$\sqrt{+a^2}$ Postulate No. 2. The equations a + b = b and ab = ba express what is called the law of commutation in logical addition and multiplication, whereby the logical sum and product do not depend on the order of the operations: we receive the same result when we add the concept b to the concept a as when we add the concept a to the concept b; similarly in the case of multiplication.

Postulate No. 3. Formulae Nos. 3^a and 3^b express what is known as the law of distribution. Equation 3^a, a(b + c) = ab + ac, is known in ordinary algebra. In the logic of algebra it has the following significance: the greatest common element of the concept a and of the concept b + c consists of two parts: the common element of the concept a and of the concept b, as also the common element of the concept a and of the concept c (and vice versa). This common element of a and b + c has therefore here been divided into two parts, hence the name given to the law of distribution (in this case the distribution of logical multiplication). In logical addition, equation 3^a, i.e., a + bc = (a + b)(a + c) applies. This means: in order to add to a the logical product b.c, it is necessary to add b to a, then c and to multiply these sums by each other (and vice versa).

Postulate No. 4. The equations 4^a and 4^b express the negative element a' (i.e., the one received from the element a by the operation of negation) by means of the corresponding positive element a and the elements 0 and 1. In accordance with these formulae, the element a' is such an element which supplements the element a to 1 (a + a' = 1), and which has the minimum community with the element a(a.a' = 0).

A few more explanations must now be added to the above preliminary information on the system of axioms of the algebra of logic. Attention

is first drawn to the fact that the elements a,b,c are in our understanding primarily not classes of objects falling under any given concepts, but they are the concepts a,b,c themselves - mental comprehensions. Thus,the concept "Man" means here not a "class of man" but "the totality of the features common to all mankind". In short,the above system of axioms is above all one of the postulates of the logic of comprehension and not of the logic of extension (class).

Let us now pass to a certain characteristic feature of this system of postulates. All the postulates appear in it in two forms; they split into two,and express themselves in twofold manner: the first equation refers to addition,the second to multiplication. We therefore have to deal with what is known as the duality of logical addition and multiplication. This duality is indicative of the unusual harmony reigning in the logical world and is expressed by this correspondence of the formulae for logical addition and multiplication. The law of duality affects, as we have seen,all the foregoing postulates of the algebra of logic. It permits us,in the case of expressions of the type of our postulates, to pass from a given formula to one dual with it simply by changing "+" for a "x",and the 1 for a 0,and conversely,as in the case of equations 1 - 4.

It will have been noticed that the symbol "$<$" does not figure among the above postulates although it signifies the relation of inclusion (implication) which is so important in logic. This relation (the inclusion of a in b),however,can be reduced to the relation "=" and to the logical operations included in the above postulates,so that

$$a < b = (b = a + b) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (I^a)$$

This definition of the relation of inclusion becomes more obvious when we take it into account that if comprehension a is contained in that of b,then addition to b of comprehension a,already contained in the former,does not change comprehension b; thus $a + b = b$,if $a < b$,and conversely: if by addition of comprehension a to comprehension b the latter undergoes no change,this means that the added comprehension a is already contained in comprehension b.

The equivalence established above,as for that matter all logical equivalence,signifies the mututal inclusion (implication) of the members of such equivalence; thus when $(a < b)$ it follows that $(b = a + b)$ and vice versa. This essential feature of all equivalence,the fact that it consists in the mutual implication of its members,is expressed by the following definition:

$$(a = b) = (a < b) + (b < a) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (II)$$

In connexion with the relation $<$ now under examination, attention is drawn to the fact that the law of duality applies to formulae expressing the equivalence between elements connected by the operation "+" and "x", as also to formulae expressing inequalities ($<$) between such elements. Thus, for instance, the formula $ab < a$ corresponds to $a < a + b$ and expresses the indubitable truth that the greatest common part of the elements a and b is contained in the element a. Hence, in order to pass from the formula for inequality to the dual one, it is not only necessary to change the "+" for a "x" and vice versa (or the 0 for 1, and vice versa) but also to transpose the members of the relation of implication. By so doing, we pass, for example, from the formula $0 < a$ to the corresponding dual one: $a < 1$.

The postulates of algebra of logic and the questions most closely connected with it should now be clear and we can pass to the most important matter in this work, viz., to the establishment of a system of logico-geometrical co-ordinates which will make it possible to geometrize the algebra of logic.

The fact that logical relations, primarily the relations of the logic of extension, permit of spatial schematization, is known to all who have studied logic and applied Euler's circles. And it is in the possibilities of these Eulerian diagrams (or of similar schemes) that lies the essential point of the problem to be attacked. Instead of using the circular diagrams which have gained such popularity since Euler began to use them (they were known, of course, before), we shall, however, consider the simpler straight-line diagrams which Leibnitz preferred to apply. Thus, if we desire to show that the two classes a and b have a common part ab, we usually take two segments (a and b) of the same straight line in such wise that they partially overlap. The part common to both represents class (ab), common to both classes, their logical product.

Such a schematization as the above (in which a and b, as also ab are represented by straight lines) has the fundamental fault that it does not express the fact that ab is a derivative in relation to a and b, and therefore belongs as it were, to another generation, to another dimension. In the above scheme, ab is represented by means of a straight line in the same way as the elements a and b which go to form it. Should we desire te depict in a spatial scheme this undubitable heterogeneity of dimension of the species on the one hand, and of the genus and speci-

fic difference on the other, the classes *a* and *b* would have to be depicted in the form of two intersecting straight lines *a* and *b*: the point of intersection would then symbolize the class ab, contained both in *a* and in *b*; then this class product would be of a dimension other than that of the classes which formed it.

In the present work, however, we are primarily concerned with the logic of comprehension and not of extension[1]. The query arises: Can two intersecting straight lines (e.g., at right angles) likewise symbolize an operation referring to logical comprehensions? The answer is, of course, in the positive; but the point of intersection of the straight lines will now from the comprehensional point of view, not signify here a formation common to the classes *a* and *b*, but a new comprehension, formed from the cumulation of the comprehensions *a* and *b*, corresponding to these classes. It will be the point $a + b$[2]. This point is not now contained in *a* and *b* (as in the case of multiplication of classes); the concept (comprehension), for instance, of "an equilateral right angle" is neither contained in the concept of "a right angle" nor in that of "equilateral", but, conversely, the constituting concepts of "a right angle" (*a*) and "equilateral" (*b*) are contained in the derivative concept of "an equilateral right angle" $(a + b)$. In other words $a < a + b$, $b < a + b$. The concepts *a* and *b*, taken (separately) as logical comprehensions are only a possibility of the sum $a + b$, are something less ~~definite~~ determinate, less definite, than the sum $a + b$. In the present schematization of a logical concept, the point $a + b$ will represent a notion fundamentally richer, a notion which is more differentiated and specified (species) than the straight lines *a* and *b* (genus and specific difference); the point $a + b$ will not here be an element of the straight line *a* or *b*, but their determination, that which the straight lines *a* and *b* constitute - in which they are contained. It is in such wise that the spatial relation of the straight lines *a* and *b* to the point $a + b$ should be understood: these lines pass through it, and they exist in it since it is their synthetic formation - and in this sense they are included within it (e.g., the lines of the pencil are included in the point-vertex of the pencil).

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[1] The spatial depiction of the comprehension of logic is more favourable from the logico-philosophical point of view; if, however, it is desired only to attain a logico-mathematical viewpoint, the spatial image of logic can be likewise considered to be a depiction of the logic of extension.

[2] The reader's attention is drawn to the fact that the symbol + (and not x) denotes "and" here and subsequently herein.

It now remains to elucidate how we are to schematize the comprehension ab,a comprehension which is more general and less determinate than the comprehensions a and b. On the basis of the above we can understand that to this proces of abstraction corresponds in the spatial field the passage from more specified elements (points) to fundamentally less specified ones (straight lines); in other words,if a and b are represented by points the scheme of the logical product ab will be the straight line joining these points. This straight line will represent what the points have in common - qualitatively the nearest element of which they are the specifications. Just as the comprehension ab,as less determinate,is implied both in comprehension a and comprehension b,so the straight line ab,as a less determinate product,will be implied in the points a and b which are more determinate than it. Hence,both in the scheme of multiplication and in that of addition of comprehensions, the straight line is implied in the points. Let the straight lines a and b intersect at right angles and the points a and b on these lines be joined by means of a straight line; the scheme thus yielded will represent spatially both the multiplication and the addition of concepts-comprehensions (see Fig. 1).

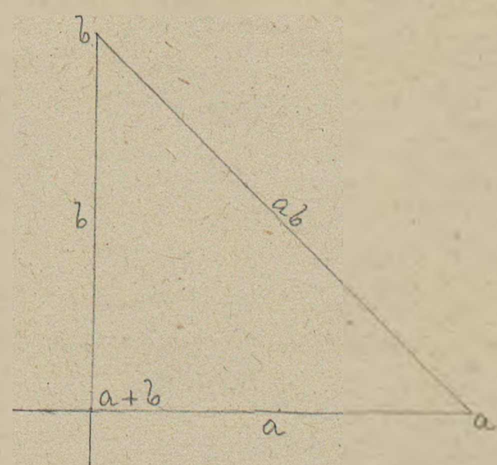

Fig. 1.

We have,in this case,a spatial representation of the relations:

$$a < a + b, \quad b < a + b$$

and of $\quad ab < a, \quad ab < b.$

This furnishes us with a clear instance of the basic duality of logistic formulae and we can already perceive its connexion with geometrical fundamental (planimetrical) duality,expressed by the fact

that the two straight lines a and b determine the point (a + b), while the two points a and b determine the straight line ab. We thus see here an absolute correspondence between the two operations of logic (addition and multiplication) and the two operations of projective geometry (section and projection).

This duality, already evident in the works of Desargues (1593-1662) - the father of modern synthetic geometry - was definitively introduced into the field of projective geometry by Poncelet (1822) and Gergonne (1826) as duality of section and projection, and into the field of exact logic by Peirce (1867), one of the founders of modern logic, as also independently of him, by Schröder (1877) as the duality of logical addition and multiplication. This striking coincidence between the properties of logical and geometrical operations is trustworthy evidence that there is a deep-lying correspondence between the world of logic and that of geometry; this coincidence has impelled us to give geometrical form to the whole of algebraic logic, one able to bring out this duality into the highest relief. It should in fact do even more. The geometrization of logic reveals duality where it would have remained concealed for long were the algebraic method solely to be applied. We have in mind the duality of elements: a, b, a', b', each of which appears in the geometrical field in a dual form: as the point a and as the straight line a. And it is this geometrical duality which has convinced us of the duality of each simple element and bids us seek for this duality the corresponding algebraic one.

Fig. 1 leads us to the thought that the point a + b is a point determined by the co-ordinates a and b, and that the points a and b will thus correspondingly be points on the axis of co-ordinates. We therefore trace out a right-angle system of co-ordinates commencing at the point O, and with the axes Oa and Ob running out from this point O. The triangle shown in Fig. 1 is then placed within this system of co-ordinates (Fig. 2).

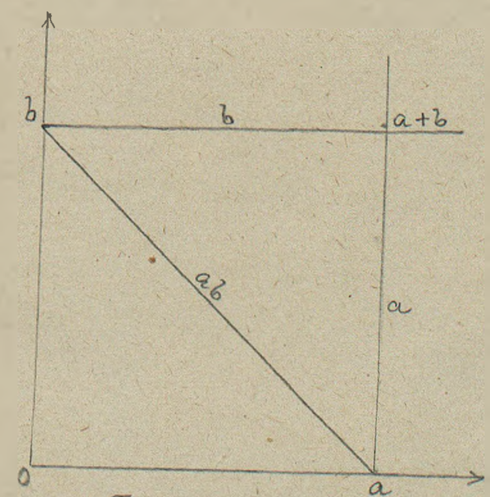

Fig. 2.

In such wise, the category of species a + b (whose constituent elements or co-ordinates, will be categories of genus a, and of specific difference b) attains geometrical form together with its logical co-ordinates. The elements a and b will now be the logico-geometrical co-ordinates. Naturally, these co-ordinates - the straight lines a and b - do not denote quantity in this case, but possess only purely qualitative, projective or directional significance.

We thus arrive at the beginnings of a two-dimensional system of logico-geometrical co-ordinates. It will now suffice to introduce negative directions, i.e. to supplement the axes of co-ordinates in Fig. 2, to extend them in the opposite directions from the origin 0 of the co-ordinates (and to trace the diagonal of the external square), in order to complete the two-dimensional system of logico-geometrical co-ordinates. There should be no difficulty in further extending this system to embrace the third dimension.

But let us dwell for a while at this plane (two-dimensional) system of logico-geometrical co-ordinates, geometrically constituting the two-dimensional logical space.

The scheme of such two-dimensional logical space is shown in Fig. 3[1].

We now pass to an examination of the geometrical correspondences of the axioms of algebraic logic, that is to say, to the constitution proper of geometrical logic.

Axiom No. 1[a] proclaims the existence of such an element 0 that a + 0 = a. Let us examine this element 0, that of the smallest compre-

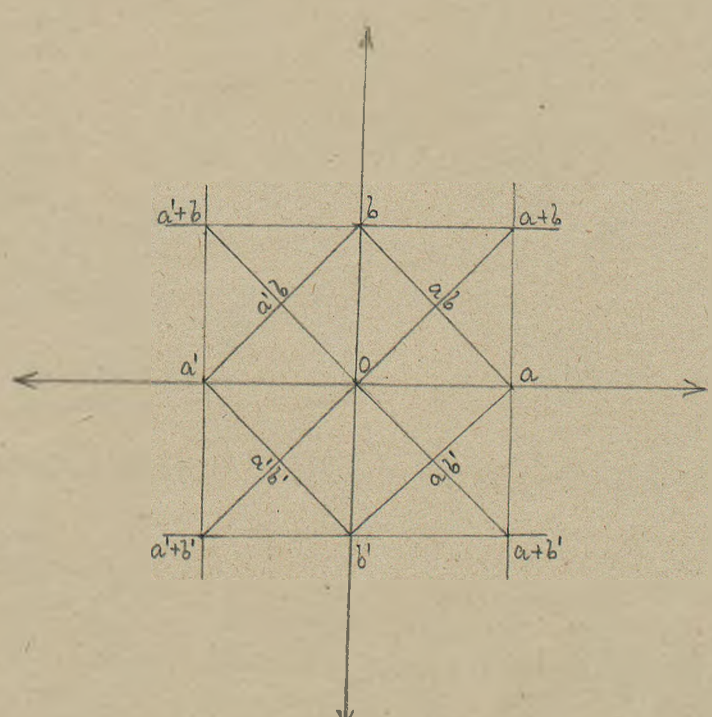

Fig. 3.

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[1] It should be borne in mind, however, that this two-dimensional logical space is the representation of only one specification of mathematical panlogic, that which has the most developed form (cf. Chapter VIII).

hension, within the above two-dimensional system of geometrical logic. If $a + 0 = a$, then, in accordance with definition I^a (p. 9) this is equivalent to the proposition that $0 < a$, i.e. 0 is smaller in comprehension than the comprehension a; in other words, it is the minimum of comprehension. This element will be represented as a plane passing through the axes of co-ordinates; it will be the two-dimensional logical space itself, that is to say, it will be something which is quite undifferentiated; it is geometrically likewise poorest in comprehension but none the less makes possible the existence of all richer forms of geometrical comprehension, and is included in such richer comprehension. Just as with the logical concept of "object" (0), it is included within every logical comprehension already differentiated, and at the same time makes it possible. Every geometrical element, therefore, being a spatial one, already contains this spatial element within itself, and the statement $a + 0$ (the geometrical element a, possessing the spatial feature 0) is naturally equivalent to saying simply a. The space factor as such, adds nothing to the comprehension of the geometrico-spatial element (since it is already contained in its comprehension) - this is the geometrical significance of axiom 1^a, $a + 0 = a$. We thus have an exact analogy to the purely logical meaning of this axiom, with one difference: that we are now dealing with the space factor in general, whilst we had previously dealt with the "object" in general. But if we are to deal with 0 already specified in the shape of the zero axis $0_{aa'}$ or $0_{bb'}$, the axiom ($a + 0 = a$) will signify geometrically that the intersection of the straight line a with the axis $0_{aa'}$ determines the point a, as has been shown in Fig. 3. The same holds good for the axis $0_{bb'}$ and the equation $b + 0 = b$. The zero axes $0_{aa'}$, and $0_{bb'}$ join up at the origin of the coordinates, i.e., at the point 0.

Axiom No. 1^b, dual to the former, states that there exists such an element 1,[1] that $a \times 1 = a$.

Passing at once to unities (1), dual as regards $0_{aa'}$ and $0_{bb'}$, i.e., to $1_{a+a'}$ and $1_{b+b'}$, we can easily represent them in the form of points at infinity upon the axes $0_{bb'}$ and $0_{aa'}$; these will be the points of intersection of the straight lines a and a' ($a + a' = 1_{a+a'}$) and of b and b'

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[1] This is of course the maximal element; for if with every element a it possesses a common part a, this signifies that every element a is included in it ($a < 1$) (Cf. p. V. Proposition 12^b).

$(b + b' = 1_{b+b'})$.[1] A glance at Fig. 3 suffices to convince that the straight line joining point a with the point $1_{a+a'}$ is actually the straight line a in accordance with axiom No. 1^b $(a \times 1 = a)$. The same holds good for the element b.

Apart from axiom No. 1^b,axiom No. 4 is most closely connected with the limitary concepts 1 and 0; we shall hence immediately pass to an examination of its geometrical significance,leaving axioms 2 and 3 to be examined later.

Axiom No. 4 affirms the existence of a negative element (a') in respect to element a and defines the former as one which,added to a, yields the element 1 as the sum (viz.,$a + a' = 1$),but multiplied by a yields the element 0 (viz.,$a \, . \, a' = 0$). These definitions of the negative element can be easily depicted. We know that the straight lines a and a' intersect at the point $1_{a+a'}$(a point at infinity on the axis $O_{bb'}$); on the other hand,we see from Fig. 3 that the axis $O_{aa'}$ is in effect the common basis for the elements a and a' (since it joins these elements by multiplication). The deeper meaning of this definition of the negative element with the help of a,0 and 1,geometrically and logically,will became apparent later - when the harmonic elements of geometrical logic will come up for examination.[2]

We can now pass to the geometrical representation of axiom No. 2, which expresses what is known as the law of commutation for components and factors: $a + b = b + a$ (No. 2^a) and $ab = ba$ (No. 2^b).

Axiom No. 2^acan be tested immediately,as it is obvious that one and the same point is determined,independently of the fact whether the line a is cut by the line b (the point a + b) or whether line b is cut by the line a (point b + a). The same holds good for the commutation of factors.

Finally we come to the axioms of distribution; $a + bc = (a + b)(a + c)$ as axiom No. 3^a,and $a(b + c) = ab + ac$ as axiom No. 3^b. Three elements appear in these axioms (a,b,and c) and for this reason these axioms in their complete form issue beyond the scope of two-dimensional (bi-elemental) geometrical logic,to which these remarks are at present restricted. Every one of the elements a,b and c represents a separate dimension,a different category,and for this reason formulae containing a positive or a negative element a and b can be depicted on a plane,

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1) Points at infinity are known in projective geometry as improper points. Similarly a straight line at infinity is known as an improper straight line. In effect,these are limitary,or trans-spatial elements.
2) Cf. footnote 2) p. 62.

whilst the tri-elemental ones must be shown in three-dimensional space. We shall therefore examine the geometrical representation of axioms No. 3 in their complete, tri-elemental form upon another page, when we come to deal with the constitution of three-dimensional logico-geometrical space (cf. pp. and). An attempt will, however, be made here geometrically to depict axioms No. 3 reduced to plane conditions by substitution of element b' for the element c. None the less, this change causes us to leave the domain of axioms and brings us to that of derived propositions. The most direct conclusions yielded by axioms No. 3 will therefore be examined in the following chapter, devoted to the geometrization of derived propositions.

Anticipating the geometrization of the tri-elemental axiom No. 3, and basing ourselves on the spatial schematization of axioms Nos. 1, 2, 4, we can confidently assert that all the derived laws of the algebra of logic, based as they are on axioms Nos. 1 - 4, will likewise achieve spatial representation.

CHAPTER II.

Geometrization of the Theorems of Algebraic Logic

The following conclusions will now be first deduced from axioms No. 3. Let us substitute the element b' for the element c in the formulae 3^a and 3^b. This gives us the law of dichotomy, a most important one in logic. We now have:

$$a + bb' = (a + b)(a + b')$$

$$a(b + b') = ab + ab'$$

As $bb' = 0$ (axiom No.4^b), whilst $a + 0 = a$ (axiom No.1^a), and correspondingly dually for 1, two dual equations for the principles of dichotomy are yielded as follows:

$$a = (a + b)(a + b') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5^a)$$

$$a = ab + ab' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5^b)$$

For example: in the case of 5^a, "a man" (a) is that which has the maximum in common with "a good man" (a + b) and "a not-good man" (a + b').

This law as a principle of geometrical logic is remarkably lucid (cf. Fig.3). Equation 5^a represents the straight line a as the logico-geometrical product of its points $a + b$ and $a + b'$. The straight line a is that which the points $a + b$ and $a + b'$ have in common; it is a genus (a) common to both species of the genus: $a + b$ and $a + b'$. The genus a has been divided into these two species - hence the name of the principle of dichotomy. Similarly with the dual formula 5^b. The only difference is that instead of the division of the straight line into two punctal elements, we have to do with the dual division of the point a into the two linear elements ab and ab'. We see how these straight lines ab and ab' are in effect united at the point a from which they issue. The duality of dichotomic formulae is revealed here in very clear fashion; we see, too, that the element a likewise changes its form when we pass from formula 5^a to formula 5^b: it is a straight line in the former and a point in the latter.

We shall now introduce the principle of tautology which best expresses the non-quantitative character of logic; it affirms;

$$a + a = a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6^a)$$

$$a.a = a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6^b)$$

These equations will be proved on the basis of axioms Nos. 1,3 and 4; theorem 6^a is based on axioms Nos. 1,3^a and 4, and theorem 6^b on a-

xioms Nos. $1, 3^b$ and 4.

$$a + a \overset{1)}{=} (a + a) \times 1 \overset{2)}{=} (a + a)(a + a') \overset{3)}{=} a + aa' \overset{4)}{=} a + 0 \overset{5)}{=} a$$

$$aa = aa + 0 = aa + aa' = a(a + a') = a \times 1 = a$$

The special significance of the theorem of tautology is obvious: the straight line a, intersected by itself, does not change. Similarly with the point a when it is joined to itself.

We now pass to the law of absorption, expressed by the equations:

$$a + ab = a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7^a)$$

$$a.(a + b) = a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7^b)$$

The proofs of these dual equations are based on axioms Nos. 1,3,4,2 and on theorem 6.

$$a + ab = a.1 + ab = a(1 + b) = a(b + b' + b) = a(b + b + b' = a(b + b') =$$
$$= a.1 = a$$

$$a(a + b) = (a + 0)(a + b) = a + 0.b = a + bb'.b = a + bbb' = a + bb' =$$
$$= a + 0 = a$$

A glance at Fig. 3 shows that it yields the spatial representation of these formulae. On the one hand we see that the straight lines a and ab actually intersect at the point a (i.e., $a + ab = a$); on the other hand, dually, point a and point $a + b$ are joined by the straight line a, i.e., $a(a + b) = a$.

In turn we pass to what is known as the development of 1 and 0. The formulae will be proved on the basis of axiom No. 4 and of the principle of dichotomy.

Axiom No.4: $1 = a + a'$. Substituting $ab + ab'$ in place of a, and $a'b + a'b'$ in place of a', in accordance with 5^b, we receive:

$$1 = ab + ab' + a'b + a'b' \quad \ldots\ldots\ldots\ldots\ldots\ldots (8^a)$$

Similarly, dually, we receive:

$$0 = (a + b)(a + b')(a' + b)(a' + b') \ldots\ldots\ldots\ldots\ldots\ldots (8^b)$$

substituting in equation 4^b ($0 = aa'$) a and a' in accordance with the dichotomic formula 5^a.

We can see the spatial scheme of these developments upon the basic plane diagram (Fig. 3): the four vertices of the outer square (which has sides: a, a', b, b') are the elements of the development of 0 (zero), its factors; the four sides of the inner square (which has vertices: a, a', b, b') are the elements of the development of 1 (unity), its components. The

1) Axiom 1^b.
2) Axiom 4^a.
3) Axiom 3^a.
4) Axiom 4^b.
5) Axiom 1^a.

dual elements of the above dual squares give us a remarkably simple and satisfactory spatial representation of the dual developments of 0 and 1.

Having become acquainted with the equations and schemes of development of 0 and 1, we can now pass to de Morgan's exceedingly important formulae - those which govern the problem of the negation of concepts. These formulae are:

$$(a + b)' = a'b' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9^a)$$

$$(ab)' = a' + b' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9^b)$$

In other words, the negation of the sums of two concepts is identical with the product of their negation (9^a) and the negation of the product of two concepts is identical with the sum of their negations (9^b). (For instance, the negation of the concept "good and wise" is equivalent to stating "either not-good or not-wise", and it must be borne in mind that the words "either ..... or" have in the algebra of logic no excluding significance; thus "either not-good or not-wise" may likewise be applied to him who is simultaneously "not-good and not-wise").

The above equations can be easily deduced from those for the development of 1 and 0, bearing postulate No.4 in mind. We shall demonstrate the validity of equation 9^b.

Postulate 4^a states that $1 = a + a'$, and 4^b that $0 = aa'$. This means that a given element and its negation yield 1 as the sum, and 0 as the product. Since this is so, taking in the equation 8^a ab as one component, we find that its negation is:

$$ab' + a'b + a'b'$$

(since also 4^b is confirmed in this case, viz., $ab(ab' + a'b + a'b') = 0$) hence:

$$(ab)' = ab' + a'b + a'b'$$

But $ab' + a'b + a'b' = a' + b'$; this can be demonstrated by developing $a' + b'$ according to equations 5^b. Namely:

$$a' = a'b + a'b', \quad b' = b'a + b'a'$$

therefore $\quad a' + b' = a'b + a'b' + ab' + a'b' = ab' + a'b + a'b'$

Hence $\quad (ab)' = a' + b'$.

In similar manner we can demonstrate dually the validity of the equation $(a + b)' = a'b'$ (9^a).

Spatially the last-named equation signifies: the negation of the vertex of the outer square is the side of the inner square in the opposite quarter across. Equation 9^b is explained spatially, thus: the negation of the side of the inner square is the vertex of the outer one in the opposite quarter across. In such wise we can at once read eight of

de Morgan's formulae from Fig.3: the four negations of the sums and the four equations of the negation of the product:

$$\left.\begin{aligned}(a+b)' &= a'b'\\ (a'+b)' &= ab'\\ (a'+b')' &= ab\\ (a+b')' &= a'b\end{aligned}\right\} 9^A \qquad \left.\begin{aligned}(ab)' &= a'+b'\\ (a'b)' &= a+b'\\ (a'b')' &= a+b\\ (ab')' &= a'+b\end{aligned}\right\} 9^B$$

The proof of each of these equations can be followed geometrically step by step, with equal facility. Let us take the equation: $(ab)' = a' + b'$. We begin with the spatial depiction of the development of 1 in the shape of the sum of the four sides of the inner square. The negation of one of these sides $(ab)'$ will appear geometrically as the total of the remaining three sides, thus: $a'b + a'b' + ab'$. But we see from Fig.3 that $a'b + a'b'$ yield $\underline{a'}$, whilst $a'b' + ab'$ yield $\underline{b'}$, so that the sum of three straight lines $a'b + a'b' + ab'$ or $(ab)'$ is contracted to the point $a' + b'$. Similarly with the other cases. It would be indeed difficult to find a more complete realization of Leibnitz's dream that every step in abstract reasoning find its spatial analogy; in very fact, we see here that "scriptura philosophica posset etiam exhiberi per linearum ductum, seu geometriam". (Gerh. Phil., VII, 41).

We shall finally examine a few of the most important theorems in which the relation of inclusion appears.

We have defined this relation (p.9) as follows:

$$(a < b) = (b = a + b) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (I^a)$$

It is easy to demonstrate that if $b = a + b$, then $ab = a$.

Namely, substituting in ab for $\underline{b}$ its equivalent $a + b$, in accordance with the condition, we receive:

$$ab = a(a + b) = a$$

And conversely, if $ab = a$, then $a + b = b$.

Namely, substituting in $a + b$ for $\underline{a}$ its equivalent ab, we receive:

$$a + b = ab + b = b.$$

We thus see that the equation $b = a + b$ is here the equivalent of the equation $a = ab$. Taking the above and equation I^a into account, the relation $a < b$ can be expressed by the equation $a = ab$, i.e.,

$$a < b = (a = ab) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (I^b)$$

The relation $a < b$ can be expressed by ~~the~~ yet another equivalent equation. Namely if $a = ab$, then

$$ab' = ab.b' = a.0 = a.a.a' = aa' = 0$$

← And conversely: if $ab' = 0$, then $a = ab$. Namely, by the principle of dichotomy, we have $a = ab + ab'$ (5^b). If $ab' = 0$, we receive:

$$a = ab + 0 = ab.$$

← Taking the above and definition I^b into consideration, the relation $a < b$ can be expressed by the equation $ab' = 0$, thus:

$$a < b = (ab' = 0) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (I^c)$$

Similarly, on the basis of the definition $a < b = (b = a + b)$, we receive:

$$a < b = (a' + b = 1) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (I^d)$$

We thus have a series of equivalent definitions of the relation of inclusion:

$$(a < b) = (b = a + b) = (a = ab) = (ab' = 0) = (a' + b = 1) \qquad \ldots\ldots\ldots (I)$$

Let us now briefly examine the geometrical representation of these equivalent definitions of the relation $a < b$.

The geometrical significance of the symbol of inclusion consists, as we know, in the passage of a straight line, a less determinate element, through a point, a more determinate element. Formula I^a: $a < b = (b=a+b)$, informs, us that the straight line a passes through the point b only when the point $a + b$ becomes spatially identical with the point b; this dependence can actually be seen on Fig.3, since the straight line a passes through the point $a + b$, and it is only when this point is shifted to point b (or point b to point $a + b$) that the straight line a can simultaneously be traced through point b. Then, however, the straight line a will simultaneously occupy the position of the straight line ab (or conversely the straight line ab will occupy the position of the straight line a), in such wise controlling the equation I^b: $a < b = (a = ab)$. The equation I^c: $a < b = (ab' = 0)$, reduced to equation $(a = ab) = (ab' = 0)$, can be represented ad oculos, by drawing attention to the fact that if the straight line ab is revolved around the point a until it coincides with the straight line a, the straight line ab' (perpendicular to ab) will automatically occupy the position of the axis $O_{\alpha\alpha'}$. And conversely, when the straight line ab' occupies the position of $O_{\alpha\alpha'}$, the straight line ab occupies the position of the straight line a. The spatial representation of the expression I^d: $a < b = (a' + b = 1)$ is effected in similar fashion.

We can now pass to what is known as the principle of simplification, expressed by the following formulae, with which we are already acquainted:

$$a < a + b \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10^a)$$

$$ab < a \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10^b)$$

This theorem can be proved as follows, by the definition of the relation of inclusion (I^b) and by the principle of absorption (7).

$$a < b = (a = ab) \quad (I^b)$$

Substituting the element $a + b$ for b, we receive:

$$a < a + b = [a = a.(a + b)].$$

Since, however, $a = a(a + b)$ is always true (7^b), then the principle

$$a < a + b$$

is likewise always true.

Theorem (10^b) is proved in the same manner.

We can see all this in Fig.3: the straight line a passes through the point $a + b$, and the straight line ab through the point a. This relation of a straight line passing through a point, or of the situation of a point upon a straight line, is known in geometry as the relation of incidence.

Analogously with the preceding method, we shall now prove the principle of identity, which appears for the most part in the algebra of logic, as follows:

$$a < a \qquad \text{.............................(11)}$$

This theorem can be proved by the definition of the relation of inclusion (I^a) and by the principle of tautology (6^a).

$$a < b = (b = a + b) \qquad (I^a),$$

substituting b for a, we receive

$$a < a = (a = a + a).$$

In view of the fact that $a = a + a$ is always true (6^a), the principle $a < a$ is likewise always true.

Theorem (11) can be also written as follows on the strength of the definition of equivalency II (p.9):

$$a = a \qquad \text{.............................(11')}$$

We can perceive the corresponding spatial representation of theorem 11 ($a < a$) on Fig.3: the straight line a passes through the point a.

Similarly, as theorems 10, on the basis of the relation of inclusion and of axioms No.1, we can prove the theorems characterizing 0 and 1 as the logical minimum and maximum:

$$0 < a \qquad \text{.............................}(12^a)$$

$$a < 1 \qquad \text{.............................}(12^b)$$

Geometrically this means: the axis $0_{aa'}$ passes through the point a; and, dually, the straight line a passes through the point at infinity $1_{a+a'}$.

x x

x

We have now listed and spatially represented a number of the most important theorems in two-elemental logic. Whenever the need may arise

for further theorems in the field of two-dimensional logic, these will be given separately at the appropriate place; we shall now, therefore pass to the constitution of tri-elemental, i.e. three-dimensional, logical space.

The point of issue of our considerations is the positive quarter (a + b) of the zero plane of two-elemental topologic (geometrical logic) with its semi-axes O_a and O_b which intersect at the origin 0 of the co-ordinates. A third axis (really a semi-axis) O_c of logico-geometrical co-ordinates is traced perpendicularly to this plane at point O; the point c is determined upon this new axis and will serve to depict the category "individual difference" (just as we had on the axis a a point a which depicted the category "genus", and the point b on axis b which depicted the category "specific difference"). We found on the plane the geometrical representation of the sum (a + b) and of the product (ab) of the plane co-ordinates a and b. We now desire to represent the sum and product of the co-ordinates a, b, c, and must therefore pass from the spatial representation of a + b and ab to the spatial representation of a + b + c and abc. In order to represent spatially the logical sum a + b + c (the category of "individual"), we trace a line perpendicular to the plane of the axes a and b at the point a + b (cf. Fig.4). This straight line will also be a + b just as in geometrical plane logic the line perpendicular to the axis at the point a was also the straight line a. We then, through point c of the semi-axis c, trace a plane parallel to the plane of the axes a and b and determine the point of intersection of this plane with the straight line a + b. This will be the point a + b + c, the co-ordinates of which are the straight lines a, b and c. In order to represent spatially the logical product abc, we draw a plane through the straight line ab and the point c, one which cuts the plane of co-ordinates along the straight lines ab, bc and ac, forming the triangle abc having the punctal co-ordinates a, b and c. It is this plane (or triangle) abc which spatially depicts the tri-elemental logical product (cf. Fig.4).

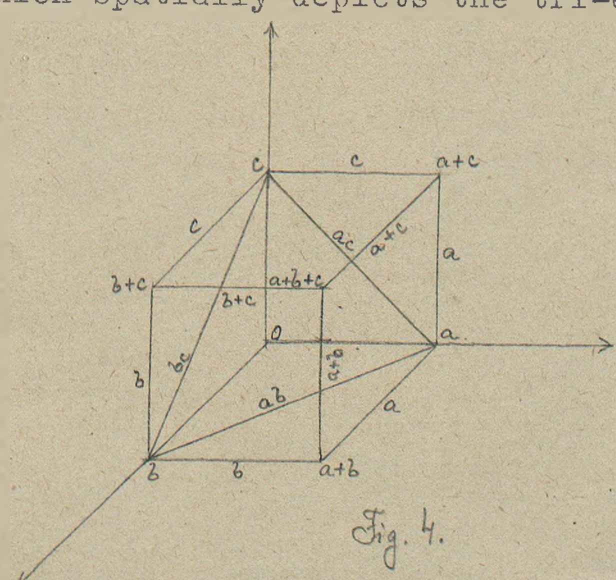

Fig. 4.

It must be borne in mind, that a plane is in this case a formation which is poorer in comprehension than a straight line, just as a straight line is poorer than a point, and that the plane thus passes through its straight lines and points (is involved within them). Bearing this in mind, we can read off the following relations from Fig. 4 direct:

$a + b < a + b + c$, and similarly: $a + c < a + b + c$ and

$b + c < a + b + c$ ....(13^a)

$abc < ab$, and similarly: $abc < ac$ and

$abc < bc$ ....(13^b)

These are of course formulae which spatially express the principle of simplification, analogously to the principle expressed for planes in formulae 10.

Formulae 13 are likewise dual ones since we pass from 13^a to 13^b by changing the "+" for a "x" and transferring the members of the relation of inclusion. We find in formulae 13 the expression of the fundamental geometrical duality which reigns in space. Namely, the straight line a+b corresponds dually to the straight line ab, whilst the point a + b + c corresponds dually to the plane abc. This is the duality for space in projective geometry: in space, the plane corresponds dually to the point, and conversely; whilst the straight line corresponds dually to the ~~strai~~ straight line.

It now suffices to introduce negative directions (i.e., to extend the axes of co-ordinates in Fig.4, tracing them out from the origin of the co-ordinates in the contrary directions) in order to receive a three-dimensional system of logico-geometrical co-ordinates, which constitute the geometrical tri-elemental logical space, a space not of species, but of individuals (cf. Fig. 5).

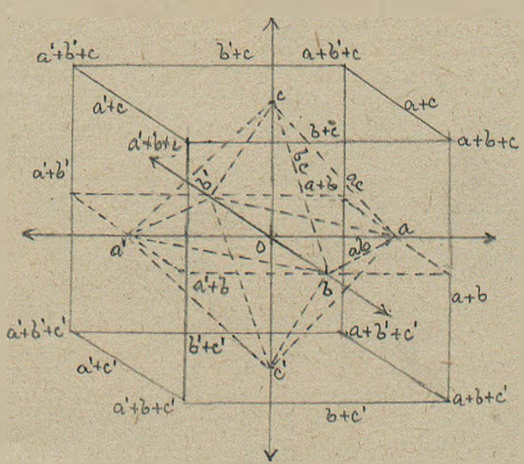

Fig. 5.

First of all, before we pass to the geometrization of the theorems of tri-elemental logic, it is necessary to carry out certain obligations which have not yet been fulfilled, viz., we must give the geometrical representation of the axiom of distribution, which in its complete form surpasses the bounds of the two-dimensional space (cf. p.17). The dual formulae of the principle of distribution are:

$$a + bc = (a + b)(a + c) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3^a)$$

$$a(b + c) = ab + ac \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3^b)$$

Let us return ti Fig.4. The left side of the equation 3^a represents the intersection of two planes: the plane a and the plane bc, which passes through the straight line bc, parallelly to the axis C_α. The plane bc (which passes through the straight lines b and c) cuts the plane a (as can be seen on the diagram) along the straight line passing through the points a + b and a + c, i.e. along the straight line (a + b).(a + c), as the right-hand side of equation (3^a) states. The intersection of the planes a and bc is therefore nothing else but the straight line (a + b)(a + c), and this is what the principle of topologic (3^a) affirms.

The dual principle 3^b states geometrically that the intersection of the planes ab and ac is the straight line a(b + c). And in fact, on examining Fig.4, it can be visually ascertained that the plane ab (passing through the straight lines a and b, and therefore through the points a + c and b + c) and the plane ac (passing through the straight lines a and c, and hence through the points a + b and b + c) have in common: the point b + c, and in addition the point a; this signifies that the intersection of the planes ab and ac (i.e., ab + ac) is the straight line a(b + c), and this is just what principle 3^b affirms.

Let us now make a closer examination of the geometrical duality of the above principles, reviewing its logical duality step by step. In equation 3^a we have on the right side the two points a + b and a + c joined by the straight line (a + b).(a + c). In dual correspondence to the above, we have the two planes ab and ac which intersect along the straight line ab + ac. Further, the left side of equation 3^a shows that the straight line (a + b).(a + c) is identical with the straight line along which the planes bc and a intersect. In accordance with this the left-hand side of equation 3^b shows dually that the straight line ab+ac is identical with the straight line which joins the points b + c and a.

Having settled these matters, we can now proceed to the spatial representations of the theorems of tri-elemental logic. We shall, however, restrict ourselves to a few of the most important examples.

First of all, there is the principle of association known to us

from ordinary algebra:

$$(a + b) + c = a + (b + c) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (14^a)$$

$$ab \times c = a \times bc \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (14^b)$$

The proof of the validity of this theorem is rather complicated and we shall therefore merely control it direct in spatial intuition. It can be seen from Fig.4: (1) that the same point (a + b + c) is yielded whether the straight line a + b is cut by plane c,or whether the straight line b + c is cut by plane a (14^a); that the same plane (abc) is yielded by the connexion of the straight line ab with the point c, as by the connexion of the straight line bc with the point a. We thus find full depictions of the logical duality of addition and multiplication in the shape of the geometrical duality of intersection and projection.

Finally,we shall furnish an exceptionally lucid exposition,from the geometrical point of view,of the principles of the development of 1 and 0,extended to three-elements,as also one of the principles of de Morgan generalized in the same manner.

The generalization of the principles of development of 1 and 0 can be attained by the dichotomic development of each element of their plane development as regards c and c' (cf. p. 19,Formula 8). We thus receive:

$$1 = abc + abc' + ab'c + ab'c' + a'bc + a'bc' + a'b'c + a'b'c' \quad \ldots\ldots (15^a)$$

$$0 = (a + b + c)(a + b + c')(a + b' + c)(a + b' + c')$$
$$(a' + b + c)(a' + b + c')(a' + b' + c)(a' + b' + c') \quad \ldots\ldots (15^b)$$

These formulae can be very simply and most satisfyingly interpreted geometrically (see Fig.5). The elements of the development of 0 are the eight point-vertices of the hexahedron having faces: a,a',b,b',c,c'; the elements of development of 1 are eight triangles forming the faces of the octahedron possessing the vertices a,a',b,b',c,c' in the middle of the faces of the above hexahedron. These two polyhedrons of Plato are - as we know - mutually dual,that is to say they have the same number of edges,and each face in one corresponds to the vertex in the other. Namely,the hexahedron has: six faces,eight vertices,and twelve edges; the octahedron has: six vertices,eight faces,and twelve edges. This geometrical duality of Plato's polyhedrons is found to be only the reflection of the logical duality of the triple developments of 0 and 1. It will be remembered that on the plane the logical duality of developments of 0 and 1 is expressed in the duality of two squares (having the sides, or vertices,a,a',b,and b'),which are found to be sections of the regular polyhedrons in question through the horizontal plane of the co-ordinates (cf. Fig.5). It would indeed be difficult to furnish a more gratifying

representation of the parallelism of thought and spatiality.

From the equations furnishing triple developments of 0 and 1, it is now but a step to the generalization of de Morgan's formulae:

$$(a + b + c)' = a'b'c' \qquad \dots\dots\dots\dots\dots\dots\dots\dots (16^a)$$

$$(abc)' = a' + b' + c' \qquad \dots\dots\dots\dots\dots\dots\dots\dots (16^b)$$

Geometrically, this means (see Fig.5): the negation (supplement) of the triangular plane-face abc is point $a' + b' + c'$ (16^b), whilst the negation of the point $a + b + c$ is the triangular face $a'b'c'$ (16^a). In other words, the negation of the faces of Plato's octahedron is the vertex of the hexahedron (dual to it and embracing it), the vertex situated in the opposite spatial octant; and conversely, the negation of the vertex of the hexahedron is the opposite face of the octahedron contained in it. In such wise we can easily read from the diagram the eight generalized formulae of de Morgan for the negation of the sum and as many dual formulae in respect of the negation of the product.

The proof of each of these formulae can likewise be easily followed geometrically step by step. In order, for instance, to negate abc (16^b), we resolve the members of this product by means of dichotomic equations. In view of the fact that

$$a = (a + b)(a + b') = (a + b + c)(a + b + c')(a + b' + c)(a + b' + c')$$

[the product of the four vertices of the face $\underline{a}$]

$$b = (b + a)(b + a') = (b + a + c)(b + a + c')(b + a' + c)(b + a' + c')$$

[the product of the four vertices of the face $\underline{b}$]

$$c = (c + a)(c + a') = (c + a + b)(c + a + b')(c + a' + b)(c + a' + b')$$

[the product of the four vertices of the face $\underline{c}$]

we receive: $abc = (a + b + c)(a + b + c')(a + b' + c)(a + b' + c')$
$(a' + b + c)(a' + b + c')(a' + b' + c)$.

The members of the above product represent seven vertices of the hexahedron. If then, the above product abc (the triangular face of the octahedron) are negated, we of course receive the point $a' + b' + c'$, the eighth vertex of the hexahedron which lacked for 1.

The foregoing examples suffice to indicate the spatial status of the principles of tri-elemental exact logic. In such manner, the foundations for an exact geometrical logic have been laid down.

CHAPTER III.

The Logical Plane and Space, and their Elements

The sytem of geometrical logic set up in the preceding chapter is markedly categorial in character and not multitudinous. A logico-geometrical plane as likewise a three-dimensional space have been reduced to quite a small number of elements, which as a matter of course acquire the significance of typical or categorial elements. In order to attain a closer knowledge of this categorial character of logico-geometrical space, the following supplementary data are introduced into the two-dimensional diagram, restricting our examinations to the plane.

We extend the straight lines b and b' and the straight lines ab and ab' from Fig.3 (see Fig.6). The straight line b will intersect ab'

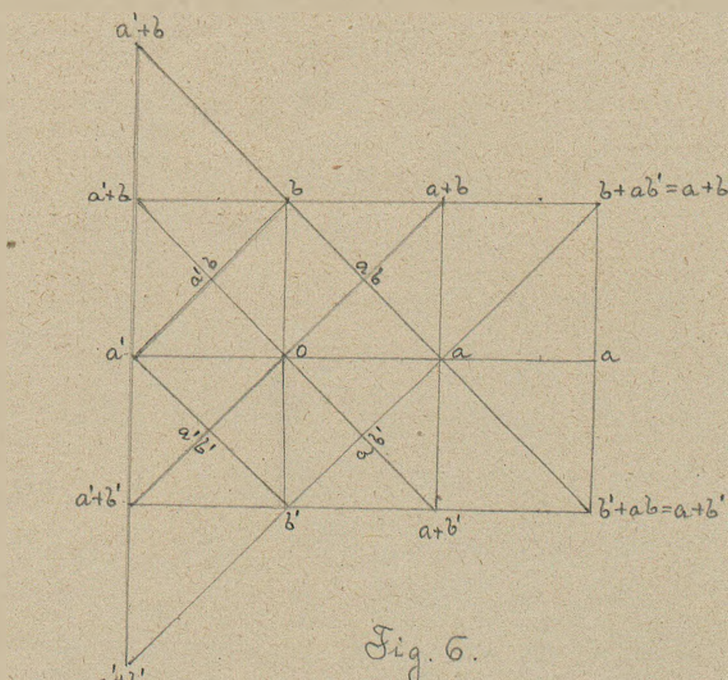

Fig. 6.

at the point b + ab' ; and the straight line b' will intersect the ab at the point b' + ab. But

$$b' + ab = a + b' \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

The proof of this can be easily furnished by substituting in the formula for distribution (see 3^a)

$$ab + c = (a + c)(b + c),$$

b' instead of c, we then receive

$$ab + b' = (a + b')(b + b') = a + b'$$

Similarly: $ab' + b = a + b$.

In such wise the straight line joining these points of intersection will be $(a + b)(a + b') = a$, and the point of its intersection with the horizontal axis will likewise be $\underline{a}$. We thus see that all the straight lines parallel to $\underline{a}$ in the right-hand half of the plane will always be straight lines $\underline{a}$, all the points on the right-hand semi-axis will likewise be points $\underline{a}$, all the points on the straight line $\underline{b}$ in the right-hand half of the plane will be points $a + b$, etc. Thus our logico-geometrical space is not an ordinary, multitudinous space, in which, e.g. every straight line parallel to $\underline{a}$, is another straight line, i.e. the straight line a_1, a_2, a_3, and so on, whilst every point on the horizontal axis differs from the other point a_1, a_2, a_3, and so on, but it is a space where the straight line $\underline{a}$ and point $\underline{a}$ (analogously in other cases too) represent all "similar" straight lines and points, and are types, forms, categories of such objects. Hence the striking quality of logico-geometrical space that it can be represented in such unusually simple, uncomplicated, concise and "economical" form. This is explained by the fact that we are dealing not with a ~~categorical space~~ numerical but with a categorical space. The distance of the various points and straight lines from the origin of the co-ordinates disappears - that distance which evokes the differences between a_1 and a_2, which doubles and multiplies them; what remains is only a type, a qualitative category, e.g. a vertical straight line in the right-hand side of the plane. The repetition of these categorial elements will in the categorial space merely be a reflection of the initial model; these reflections cannot in their pseudo-separate form be considered to be separate categories (e.g., the second point on the straight line $\underline{a}$, or the second point $a + b$ on the horizontal straight line $\underline{b}$). This fundamental trait of categoriality naturally passes from logico-geometrical space to a science which investigates this space, i.e., to geometrical logic, which it likewise fully characterizes.

Categorial-logical space has a markedly architectonic character, a structural one which immediately strikes us when we come to look at it. This character is due to two reasons: first of all to the duality reigning supreme in logico-geometrical space, then to the situation of its positive and negative elements. This architectonic character of the logical world has been revealed and visualized by the spatialization of algebraic logic effected above, and the closer examination of logico-spatial architectonics will form the subject of the following chapters.

We shall, however, first endeavour by means of purely geometrical

methods to determine the fundamental types - the fundamental categories - which can be seen in logico-geometrical space. And here, at the very foundations of geometrical logic, we shall once again find proof of that fundamental correspondence between the non-spatial world of logic and the spatial world of geometry.

We have a categorial-geometrical plane with its two fundamental elements: a point and a straight line. The query arises: What various types, or qualities, of these points and straight lines exist in this plane? This query is understood to be tantamount to asking about the various categories of the geometrical elements (which depict the corresponding logical categories). What, however, does "quality" signify in geometrical space; in what does the qualitative difference of points or straight lines consist? The answer is fairly obvious: this difference depends on the various positions of these formations in relation to the axis of co-ordinates and the quadrants of the plane established by it; thus, some of the formations are to found only in one of these quadrants (e.g., point a + b), and others in two of them (e.g., the straight line $\underline{a}$), or even in three or in all four of these quadrants. We thus receive all the possible categories, all the possible logico-geometrical qualities on the plane, when we reply to the query as to which formations (points, straight lines) are to be found:

I. only in one of the four quadrants of the plane;
II. in two;
III. in three;
IV. in all four of the quadrants;
V. outside the quadrants of the plane.[1)]

Let us now pass to a systematic examination of each of these queries, beginning with the first.

(I^a) As regards geometrical points, the most usual and obvious example is that type ~~the very~~ of points which is located in only one of the quadrants; in fact the very possibility of other types of points appears at first sight to be problematical. These ordinary 1-quadrant points are of the type: a + β, where $\underline{a}$ is either a logico-geometrical co-ordinate $\underline{a}$ or $\underline{a'}$, whilst β is a co-ordinate $\underline{b}$ or $\underline{b'}$. In dependence on the quadrant

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1) In the case of one-dimensional space, i.e., for straight lines, these questions can be reduced to three in number. Which formations are located in one of the halves of such line, which in both, and which beyond the halves? Such formations will be four in number: two of them, each in one of the two halves of the line, are the elements $\underline{a}$ and $\underline{a'}$; the third element in both halves (really at their common boundary) is the origin of the co-ordinates 0; finally, the fourth element is the element 1, a point at infinity on the given straight line.

of the plane in which such categorial point is located, we receive four categories of points belonging to group I^a:

$$a + b, a + b', a' + b, a' + b' \text{ (cf. Fig.3 or 6)}$$

These points in geometrical logic naturally answer to the four varieties of points in analytical geometry, determined by the straight lines:

$$(x = a, y = b), (x = a, y = -b), (x = -a, y = b) \text{ and } (x = -a, y = -b).$$

(I^b) If now we pass from the points to the straight lines and search for straight lines contained in only one of the quadrants, none will be found; every unlimited straight line (but not a segment of a straight line) extends beyond one of the quadrants and is not contained within it.

(II^a) We pass in turn to the points which are supposed to exist in two quadrants of the plane. This is of course possible when these are end-points, i.e., when they are situated at the common limits of two quadrants. These end-points are primarily of type: a or β, i.e., logico-geometrical co-ordinate points. There are four end-points of this kind:

a, b, a', b'.

Point a is the end-point for quadrants I and II (right -half) on the semi-axis O_a;

Point b is the end-point for quadrants I and III (upper half) on the semi-axis O_b;

Point a' is the end-point for quadrants III and IV (left-half) on the semi-axis $O_{a'}$;

Point b' is the end-point for quadrants IV and II (lower half) on the semi-axis $O_{b'}$.

These end-points represent the category of elementary points which determine more complex formations when in combination.

To categories of end-points in geometrical logic correspond the co-ordinate points in analytical geometry:

$$(x, 0), (0, y), (-x, 0), (0, -y).$$

This is not all, however. We still have two end-points ("two quadrant" points) namely those for the opposite quadrants I and IV, and II and III. These are the points at infinity on the diagonals of the external square, passing through the afore-mentioned quadrants:

$$a'b + ab' \text{ and}$$
$$ab + a'b'.$$

We shall return to these points below in the second part of section II^b.

(II^b) If now we pass from points to straight lines and inquire respecting those straight lines which are situated in two quadrants (i.e.,

pass through two quadrants), the query resolves itself into two questions: Are there any straight lines which pass through the adjacent quadrants (e.g., I and II or I and III)? And: Are there any straight lines which pass through opposite quadrants (e.g., I and IV)?.

As regards the first type of "two-quadrant" straight lines, in accordance with the four categories of "two-quadrant" points, we find four types of such co-ordinate straight lines:

$$\underline{a},\ \underline{b},\ \underline{a'},\ \underline{b'}.$$

The straight line $\underline{a}$, in quadrants I and II (right-half) and is $\parallel$ to axis $O_{bb'}$

The straight line $\underline{b}$, in quadrants I and III (upper half) and is $\parallel$ to axis $O_{aa'}$

The straight line $\underline{a'}$, in quadrants III and IV (left-half) and is $\parallel$ to axis $O_{bb'}$

The straight line $\underline{b'}$, in quadrants IV and II (lower half) and is $\parallel$ to axis $O_{aa'}$

The geometrical fact that each of the straight lines $\underline{a}$, $\underline{b}$, $\underline{a'}$ and $\underline{b'}$ passes through two quadrants of the plane, can be expressed algebraically, presenting these lines by the dichotomic method:

$$a = (a + b)(a + b')$$
$$b = (a + b)(a' + b)$$
$$a' = (a' + b)(a' + b')$$
$$b' = (a + b')(a' + b')$$

These lines, as co-ordinate straight lines, which jointly determine formations of a higher type, are simple formations (less complex than, for instance, straight lines of the type ab). They correspond to the co-ordinate straight lines of Descartes' analytical geometry, the equations of which are:

$$x = a;\ y = b;\ x = -a;\ y = -b.$$

These are not, however, all the "two-quadrant" straight lines. There can also be straight lines which do not pass through two adjacent quadrants, but through two opposite quadrants. We have two pairs of such quadrants: I and IV, II and III, with two corresponding categorial ~~strai~~ straight lines, of which one passes through quadrants I and IV, and the other through quadrants II and III. The former joins the points $a + b$ and $a' + b'$, whilst the latter joins the points $a + b'$ and $a' + b$. They are thus the straight lines:

$$(a + b)(a' + b'), \qquad (a + b')(a' + b).$$

In view of the fact, however, that $(a + b)(a' + b') = aa' + ba' + ab' + bb' = a'b + ab'$, whilst $(a + b')(a' + b') = aa' + b'a' + ab + b'b = ab + a'b'$, the above two straight lines can be likewise represented in additive form:

$$a'b + ab', \qquad ab + a'b'.$$

The straight lines of analytical geometry passing through the origin of the co-ordinates at the same angle as regards both axes (i.e., through opposite quadrants) correspond to these two categorial straight lines in geometrical logic. Their equations are:

$$x = y \quad \text{and} \quad x = -y.$$

So far we can see no connexion between these equations and the above formulae of geometrical logic. It will suffice, however, if we recall formula I^c, which states that $a < b = ab'$ [or more precisely = $(ab' = 0)$], to make this connexion evident.

Namely: the formula: $a'b + ab'$ will then assume the form: $(b < a) + (a < b)$, or in accordance with the definition of equivalence, the form $a = b$, whilst the formula: $ab + a'b'$ then acquires the form of: $(a < b') + (b' < a)$ or $a = b'$.

Thus we see here again the correspondence between the formulae of topologic and of analytical geometry.

Resuming section II^b, we can state that we have six categories of "two-quadrant" straight lines:

(1) $a = (a + b)(a + b')$

(2) $b = (a + b)(a' + b)$

(3) $a' = (a' + b)(a' + b')$

(4) $b' = (a + b')(a' + b')$

(5) $(a + b)(a' + b')$ or $a'b + ab'$ or $a = b$

(6) $(a + b')(a' + b)$ or $ab + a'b'$ or $a = b'$

Let us now examine the relation of the straight lines (5) and (6) to the four remaining ones. We see that, in common with the first four, they are likewise connections of "single-quadrant" points, but with the difference that they constitute connections of opposite points and not of adjacent ones - hence six such straight lines exist and not four.

In accordance with this, dually, we likewise have in II^a not four points but six "two-quadrant" points. The "two quadrant" points <u>a</u>, <u>b</u>, <u>a'</u> and <u>b'</u> can be presented in dichotomic form - dually to the straight lines of the same denomination - as follows:

$$a = ab + ab'$$
$$b = ab + a'b$$
$$a' = a'b + a'b'$$
$$b' = ab' + a'b'$$

We still lack two other of the possible additive combinations of the straight lines which figure on the right-hand sides of the above equations, i.e., the combinations of the opposite sides of the inner

square, and not of the adjacent ones as in the above set of four lines. These will be the points:

$$ab + a'b' \ [= (a + b')(a' + b)] \text{ and}$$
$$a'b + ab' \ [= (a + b)(a' + b')]$$

dually to the straight lines (5) and (6).

These points, the intersections of the parallel straight lines, are at infinity upon the straight lines of the same denomination, as can be seen on Fig. 3.

III[a]. We must now find the points which might be situated in three quadrants; naturally, these would be the limitary points for three quadrants (and solely for three quadrants). But such points are impossible, as impossible as straight lines which would be situated within only one quadrant. There can be no "three-quadrant" points.

III[b]. On the other hand, straight lines passing through three quadrants belong to a very ordinary topological category: in its ordinary nature or normality, this category fully corresponds to the category of "single-quadrant" points. There can be four types of such straight lines:

The straight line ab, passing through the quadrants I, II, III
The straight line ab', passing through the quadrants I, II, IV
The straight line $a'b$, passing through the quadrants I, III, IV
The straight line $a'b'$, passing through the quadrants II, III, IV

Naturally, we are concerned here not with segments ab and so on, but with a straight line, indetermined, passing through the points a and b; such a straight line will in effect pass not only through quadrant I but also through quadrant II (lower right) and quadrant III (upper left). The query now arises whether it is possible to express analytically (algebraically) that the straight line ab passes through three quadrants (I, II and III). The answer is in the positive, when we consider that

$$ab = (a + b)(a + b')(a' + b)$$

since $$a = (a + b)(a + b'), \text{ whilst } b = (a + b)(a' + b).$$

In such wise the straight line ab can be presented in a form which shows that this line passes through the points: $a + b$, $a + b'$, $a' + b$; i.e., three categorial points of the above-mentioned quadrants. But it will be asked how the straight line ab can pass through these points, for they are situated outside this straight line, as can be seen from the representation of the topological plane. In such case, we must bear in mind that (as was demonstrated on p.) the categorial element can be multiplied; it can occupy many positions, naturally within those quadrants to which it belongs by virtue of its category (e.g., the point a,

or the straight line $\underline{a}$, or the point $a + b'$, etc. as in Fig.6). It can be seen on Fig.6 that the straight line ab in spite of all passes through the point $a + b'$, and if we consider the intersection of this straight line with the straight line $\underline{a'}$, we recieve the point $ab + a' = a' + b$; in other words, the straight line ab likewise passes through the point $a' + b$.

It now remains only to demonstrate that the straight line ab also passes through the point $a + b$. With this object in view, we consider the point of intersection of this straight line with the straight line $(a + b)(a' + b')$ or $a'b + ab'$. This point will be $ab + a'b + ab' = a+b$. Hence point $a + b$ likewise lies on the straight line ab, and this straight line actually passes through the three points: $a + b$, $a + b'$ and $a' + b$, characterizing quadrants I, II and III of the plane.

In such wise, the categorial straight lines of section III[b] can be presented in a form which would be the algebraic correspondence of their "three-quadrant" geometrical character.

Thus:

$$ab = (a + b)(a + b')(a' + b)$$
$$ab' = (a + b')(a + b)(a' + b')$$
$$a'b = (a' + b)(a' + b')(a + b)$$
$$a'b' = (a' + b')(a' + b)(a + b')$$

The formulae ab, ab', $a'b$ and $a'b'$, in their original form correspond to those of analytical geometry, which determine the straight line by means of the points $(\underline{a},\underline{b})$ at which this straight line cuts the axes of co-ordinates x and y.

We can now pass to an examination of the last two, extreme, categorial possibilities (IV and V), in respect of formations having maximal and minimal extension.

IV[a]. It must first of all be ascertained whether it is possible for a point to be situated in all four quadrants. Were such a point possible, it could only be a point limitary for all the quadrants (since it is only in significance of "presence on the boundary", that we can speak of a point being situated in more than one quadrant). Our considerations in respect of the "extension" of the elements of categorial topologic have so far shown that the richer the element, the "greater" its comprehension, i.e., the more determinate it is, the smaller its "extension" or "scope". Thus, when we have a diminishing series of elements $a + b > a > ab$ their "extension", on the contrary, grows and from "single-quadrantness" for $a + b$, through the "two-quadrantness" for "$\underline{a}$, passes to "three-quadrantness" for ab. The "four-quadrant" point

will therefore be the minimum element (point), the logical 0, and this we see at the origin of the co-ordinates, at the boundary where the four quadrants of the logical plane are contiguous with each other.

IV^b. The query now arises: Is it possible for a straight line to be present in all the quadrants? On the basis of what has just been stated, we understand that such a straight line will be the axis $aa' = 0_{aa'}$, and the axis $bb' = 0_{bb'}$.

The point a is situated in quadrants I and II, the points a' in quadrants III and IV, the straight line aa' is therefore present in ~~quadra~~ quadrants I, II, III and IV. Similarly with the straight line bb'.

This "four quadrantness" of the zero axis can be represented algebraically, developing 0 according to a, a', b and b' (see p.19).

Namely:

$$0 = (a + b)(a + b')(a' + b)(a' + b')$$

We can likewise examine the plane of the system of co-ordinates as the maximal common substrate of the axis aa' and bb' (similarly as the straight line ab is examined as that which is maximally common to points a and b), and thus $aa'bb'. = 0$ can be considered as the maximal extension, as something which is present in every one of the quadrants (cf. pp.14 and 15).

V^a. Finally, the query arises: Are there any geometrical elements - primarily, points - beyond the quadrants of the plane and not present within any of these quadrants? There are such points and these are, as we already know, elements which are maximal in comprehension, the logico-geometrical points = 1.

These will be points, dual to the axis aa' and bb', the points $a+a' = 1$ and $b + b' = 1$, points at infinity: the first on the axis $0_{bb'}$, and the second on the axis $0_{aa'}$. The straight line a passes through quadrant I and II, and the straight line a' through quadrant III and IV; their point of intersection is situated within the quadrant common to the first and second pair of quadrants[1], i.e. in none of these quadrants.

V^b. We also have a straight line situated beyond the quadrants of the plane. It is the maximal straight line, dual to the origin of the co-ordinates ($0 = 0_{aa'} + 0_{bb'}$), the straight line $1 = 1_{a+a'} \times 1_{b+b'}$. This is the straight line at infinity, joining the points at infinity $1_{a+a'}$ ($= a + a'$) and $1_{b+b'}$ ($= b + b'$).

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[1] Similarly, for instance, point a, as the point of intersection of the straight lines ab and ab', is situated within the quadrants I and II, common to the quadrants through which pass the straight line ab (I, II and III) and the straight line ab' (I, II and IV).

x x

x

Let us now resume the results secured.

All the categorial elements of the logico-algebraical plane have been indicated and classified, on the basis of their spatial qualities, viz., in dependence on the number of quadrants of the plane in which they are present. Considering that the maximum comprehension of the categorial elements is connected with the minimum of its presence, then, desiring to arrange these categorial elements in the order of their increasing presence, we must begin with the group of maximal comprehension since this will be the one of minimal presence. This can therefore be termed the zero group.

In such wise, we receive the data contained in Table I.

Table I.

Categorial Elements of the Logico-Geometrical Plane.

Plane elements present

in none of the quadrants	in one quadrant	in two quadrants	in three quadrants	in all quadrants
$1_{a+a'}$	$a + b$ (I)[1]	a(I,II) a' (III,IV)	ab (I,II,III)	$0_{aa'}$
$1_{b+b'}$	$a + b'$ (II)	b(I,III)b' (II,IV)	ab' (I,II,IV)	$0_{bb'}$
$1_{(a+a')(b+b')}$	$a' + b$ (III)	$ab + a'b'$ (II,III)	$a'b$(I,III,IV)	$0_{aa'+bb'}$
	$a' + b'$ (IV)	$ab' + a'b$(I,IV) and 6 dual elements: a, a' ; b, b' ; $ab' + a'b$; $ab + a'b'$	$a'b'$ (II,III,IV)	

If all the equivalent categorial elements (i.e. those which appear in the same quadrants of the plane) are considered only once in the table, it will present itself in simplified form (Table II).

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[1] The Roman figures refer to the quadrants in which the given element is present.

Table II.

Unequivalent Categorial Elements of the Logico-Geometrical Plane.

Unequivalent plane elements present

In none of the quadrants	in one quadrant	in two quadrants		in three quadrants	in all quadrants
0	1	2		3	4
1	$a + b$	a,	a'	ab	0
	$a + b'$	b,	b'	ab'	
	$a' + b$	$ab + a'b'$,	$ab' + a'b$	$a'b$	
	$a' + b'$			$a'b'$	

Table I covers twenty-six elements and Table II sixteen = 2^4 (the power indicates the number of quadrants of the plane or of the points representing them).

In such manner we have arrived at all the elements of categotial plane geometry, at all the possible types (forms or categories) of positions or of directions on the plane. And here categorial analysis yields as a result not only all the kinds of situation in the two-dimensional field of geometry, but also at the same time all the kinds of concepts in the two-elemental field of logic, thus testifying to the complete correspondence between the domain of thought and that of space.

These possible logico-geometrical situations have been ascertained by referring them to the four quadrants of the plane. Every one of these quadrants is represented categorially by a point of the type $a + \beta$ (Group I), where <u>a</u> is the co-ordinate <u>a</u> or <u>a'</u>, whilst β is the co-ordinate <u>b</u> or <u>b'</u>. We can thus easily arrive at the data in Table II by purely analytical means, starting from the denominations for four quadrants ($a + b$, $a + b'$, $a' + b$, and $a' + b'$) and examining all their possible combinations, 16 in number, i.e., those composed of 1,2,3,4 and of 0 elements of the above point-quadrants.

These combinations are as follows:

(1) $a + b$ +	(2) $a + b$ −	(3) $a + b$ −	(4) $a + b$ −
$a + b'$ −	$a + b'$ +	$a + b'$ −	$a + b'$ −
$a' + b$ −	$a' + b$ −	$a' + b$ +	$a' + b$ −
$a' + b'$ −	$a' + b'$ −	$a' + b'$ −	$a' + b'$ +
(5) $a + b$ +	(6) $a + b$ +	(7) $a + b$ +	(8) $a + b$ −
$a + b'$ +	$a + b'$ −	$a + b'$ −	$a + b'$ +
$a' + b$ −	$a' + b$ +	$a' + b$ −	$a' + b$ +
$a' + b'$ −	$a' + b'$ −	$a' + b'$ +	$a' + b'$ −
(9) $a + b$ −	(10) $a + b$ −	(11) $a + b$ +	(12) $a + b$ +
$a + b'$ +	$a + b'$ −	$a + b'$ +	$a + b'$ +
$a' + b$ −	$a' + b$ +	$a' + b$ +	$a' + b$ −
$a' + b'$ +	$a' + b'$ +	$a' + b'$ −	$a' + b'$ +
(13) $a + b$ +	(14) $a + b$ −	(15) $a + b$ +	(16) $a + b$ −
$a + b'$ −	$a + b'$ +	$a + b'$ +	$a + b'$ −
$a' + b$ +	$a' + b$ +	$a' + b$ +	$a' + b$ −
$a' + b'$ +	$a' + b'$ +	$a' + b'$ +	$a' + b'$ −

The plus sign which figures alongside the constitutive element (i.e.,the element: $a + b$, $a + b'$, $a' + b$, $a' + b'$) signifies that this element enters into account in the given combination. If a given combination constitutes two or more elements,the element y~~ø~~ielded by it is the logical product of the constitutive elements.[1] In other words,the combinations of constitutive elements are here disjunctive,i.e.,product-combinations. In such wise,the above combinations constitute the sixteen following unequivalent elements of two elemental logic:

(1) $a + b$ (2) $a + b'$ (3) $a' + b$ (4) $a' + b'$

(5) $a\,[= (a + b)(a + b')]$ (6) $b\,[= (a + b)(a' + b)]$

(7) $(a + b)(a' + b')$ (8) $(a + b')(a' + b)$

(9) $b'\,[= (a + b')(a' + b')]$ (10) $a'\,[= (a' + b)(a' + b')]$

(11) $ab\,[= (a + b)(a + b')(a' + b)]$

(12) $ab'\,[= (a + b)(a + b')(a' + b')]$

(13) $a'b\,[= (a + b)(a' + b)(a' + b')]$

(14) $a'b'\,[= (a + b')(a' + b)(a' + b')]$

(15) $0\,[= (a + b)(a + b')(a' + b)(a' + b')]$

(16) $1\,[= (a + b)(a + b')(a' + b)(a' + b')]' = a'b' + a'b + ab' + ab$

All these elements can be seen on our image of the two-elemental logical world,which shows us ad oculos,however,that we have more than

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[1] Or too,equivalently,it is the negation of the logical product of the elements which do not enter into the given combination (marked with a minus sign).

sixteen elements in that world, for some of them appear in various although equivalent forms. This applies primarily to the "two-quadrant" elements (5 - 10), which appear not only as the straight lines: $a, b, (a + b)(a' + b'), (a + b')(a' + b), b', a'$, but also as the points: $a, b, a'b + ab', ab + a'b', b', a'$ which are equivalents of these straight lines; it also applies to the limitary elements 0 and 1, of which each appears in three forms.[1] Namely, zero as the axis aa', the axis bb', and the origin of the co-ordinates $(aa' + bb')$, and unity as the point at infinity $a + a'$, the point at infinity $b + b'$, and the straight line at infinity $(a + a')(b + b')$.

x x

x

We shall now, however, leave two-elemental geometrical logic (or plane topologic) for a while and pass to three-dimensional geometrical logic (stereo-logic). The representation of three-dimensional "logical space" has already been given in Fig.5 (p.25); we must now endeavour to acquire a better knowledge of the elements of this categorial space by making a brief examination analogous to that made in the preceding chapter when dealing with the categorial plane.

Just as we ascertained all the possible logic-geometrical plane qualities in that case, by examining the plane logic-geometrical elements from the point of view of their presence in no, one, two, etc. quadrants of the plane, so shall we now classify all the objects of logical space from the point of view of their presence in no, one, two... eight octants of three-dimensional space. Each of these spatial octants is categorially represented by a point of type $\alpha + \beta + \gamma$, where $\underline{a}$ is a co-ordinate $\underline{a}$ or $\underline{a'}$, β a co-ordinate $\underline{b}$ or $\underline{b'}$, and γ a co-ordinate $\underline{c}$ or $\underline{c'}$.

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1) Should we determine the straight line at infinity as the product of the points lying at infinity on the slanting axes, thus as $(a'b + ab')(ab + a'b') = (a + b)(a' + b')(a + b')(a' + b) = 0$, we would receive still another, a fourth, zero form; but this zero form would now coincide spatially with unity, as a straight line at infinity, fixed by points at infinity lying on the ordinary axes. Dually should we determine the origin of the co-ordinates as the sum of the slanting axes, thus as $(a' + b)(a + b') + (a + b)(a' + b') = ab + a'b' + ab' + a'b = 1$, we would receive still another, fourth unity form; but this unity form would now coincide spatially with zero, as the point-origin of the co-ordinates fixed by the intersection of the ordinary axes.

All the possible product-combinations from these eight points will yield all the possible elements (unequivalents)[1] of three-dimensional logical space to the number of $2^{(2^3)} = 2^8 = 256$. All these analytic (logical) elements can be represented by the geometrical forms corresponding to them (see Fig.5). As we know from the theory of combination, this number of 256 elements splits up into nine groups, the first of which we shall call the zero group in view of the fact that it contains an element not found in any of the spatial octants. The distribution of these elements in the various groups is as follows:

Serial No. of Group	No. of elements
(0) Zero	1
(I) First	8
(II) Second	28[2]
(III) Third	56[2]
(IV) Fourth	70[2]
(V) Fifth	56
(VI) Sixth	28
(VII) Seventh	8
(VIII) Eighth	1

Let us now make a closer examination of some of these groups for the sake of example.

(I and VII). Group I comprises 8 elements which represent the result of combinations "of each one" of 8 fundamental elements, or, in other words, it represents just these fundamental elements:

$$a + b + c,\ a + b + c',\ a + b' + c,\ a + b' + c',\ a' + b + c,\ a' + b + c',\ a' + b' + c,\ a' + b' + c'$$

We have here the eight vertex-points of a categorial hexahedron. Each of these elements is found only in one octant of the logical space of which it is the representative.

We shall not now pass direct to Group II but shall consider the group dual to Group I, i.e. Group VII. Every element of Group I will correspond to a dual element of Group VII, and this correspondence is already presaged by the fact that the number of elements in Groups I and VII (as in II and VI, and III and V) is one and the same. Each of the eight elements of Group VII will be present in seven octants of space, and this is expressed analytically in such wise that it represents the

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[1] We shall not enter upon a closer consideration of the equivalent elements herein.

[2] $28 = \frac{8.7}{1.2}$; $56 = \frac{8.7.6}{1.2.3}$; $70 = \frac{8.7.6.5}{1.2.3.4}$

product of the seven fundamental elements.

Indeed, the element dual to $a + b + c$ i.e. the element abc is that element which appears as the product of the following seven fundamental elements:

$$(a + b + c)(a + b + c')(a + b' + c)(a + b' + c')(a' + b + c)(a' + b + c')$$
$$(a' + b' + c) \quad \text{(cf. p.28)}$$

and represents a plane passing through all the octants of the logical space but with the exception of the octant represented by the element $(a' + b' + c')$. It is this type of eight elements dual as regards the elements of Group I which constitutes Group VII:

$$abc,\ abc',\ ab'c,\ ab'c',\ a'bc,\ a'bc',\ a'b'c,\ a'b'c'.$$

These elements are planes, in accordance with the geometrica principle of duality in space by which the plane corresponds to the point and vice versa. They represent the eight faces of an octohedron dual to the above-mentioned hexahedron.

(II and VI). We now pass to Group II, which contains 28 elements which are the result of combinations "of pairs" of the eight fundamental elements. This group splits up into three sub-groups in dependence on the fact whether each of the eight fundamental points (e.g., point $a + b + c$) combines multiplicatively with the fundamental point, having only one element differing as regards the symbol ($'$) from the elements of a given point[1], or, with the fundamental point having two[2] or three[3] elements, differing in respect of the ($'$) from the elements of a given point.

The first sub-group comprises twelve straight lines[4] which are the edges of the hexahedron, thus, the straight lines $a + b$, $a + b'$, $a' + b$, $a' + b'$, $a + c, \ldots\ldots$ $b + c \ldots\ldots$ (cf. here, as usual, with Fig.5).

The second sub-group comprises twelve straight lines which are diagonals of the six faces of the hexahedron, namely, the straight lines:

$$(a + b + c)(a + b' + c') = a + (b + c)(b' + c'),\ (a + b + c)(a' + b + c'),$$
$$(a + b + c)(a' + b' + c) \quad \text{and so on.}$$

These straight lines represent projections of the slanting axes

[1] E.g., - if point $a + b + c$ - with the point $a + b + c'$, $a + b' + c$, $a' + b + c$. The corresponding combinations will be: $(a + b + c)(a + b+c') = a + b$; $(a + b + c)(a + b' + c) = a + c$; $(a + b + c)(a' + b + c) = b + c$.

[2] E.g., with the point: $a + b' + c'$, $a' + b + c'$, $a' + b' + c$. The corresponding combinations will be: $(a + b + c)(a + b' + c') = a + (b + c)(b' + c') = a + bc' + b'c$, and so on.

[3] E.g., with the point: $a' + b' + c'$. The corresponding combination will be: $(a + b + c)(a' + b' + c')$.

[4] Or twelve points of the same denomination.

$(b + c)(b' + c)$ and $(b + c').(b' + c)$ on the faces a and a', of the slanting axes $(a + c)(a' + c')$ and $(a + c')(a' + c)$ on the faces b and b', and finally of the slanting axes $(a + b)(a' + b')$ and $(a + b')(a' + b)$ on the faces c and c'.

The third sub-group comprises four diagonals of the whole hexahedron:

$$(a + b + c)(a' + b' + c'); \; (a + b' + c)(a' + b + c'); \; (a' + b + c)(a + b' + c'); \; (a' + b' + c)(a + b + c').$$

Passing now to Group VI, dual as regards the above Group II, we likewise have 28 elements divided into three sub-groups (dual in respect of the three sub-groups of Group II). They will be straight lines, passing through six octants of the space.

The first sub-group comprises twelve straight lines[1] which are the edges of an octohedron, thus, the straight lines ab[2], ab', $a'b$, $a'b'$, ac......., bc.......

The second sub-group comprises twelve straight lines, passing in pairs through six vertices of an octohedron, namely, the straight lines: $abc + ab'c' = a(bc + b'c')$, $abc + a'bc'$, $abc + a'b'c$ and so on. The second straight line, passing through the vertex a will be the straight line: $ab'c + abc' = a(b'c + bc')$, and so on.

The third sub-group comprises four straight lines at infinity which are sections of opposite (parallel) faces of the octohedron, viz.:

$$abc + a'b'c', \; ab'c + a'bc', \; a'bc + ab'c', \; a'b'c + abc'.$$

Finally, disregarding the remaining central groups, we pass to an examination of the extreme Groups VIII and 0.

Group VIII, as the product of eight fundamental elements (vertices of a hexahedron) will represent the logical 0 (the origin of the co-ordinates), whilst the group dual to it, the zero group, will represent the element dual to 0, i.e. the logical 1 (a plane at infinity). We must bear in mind, however, that the elements of Groups VIII and 0 have various forms (points, straight lines, planes) which we have not examined herein.

In such wise, we can arrive at the geometrical elements corresponding to the 256 unequivalent elements of logical space.

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1) Or twelve planes of the same denomination.

2) The straight line $ab = (a + b + c)(a + b + c')(a + b' + c)(a + b' + c')(a' + b + c)(a' + b + c')$ is present in six octants represented by the above points: $a + b + c$, $a + b + c'$...

CHAPTER IV.

The Elements of the Categorial Plane and Complete Dual Squares

Two dual squares are an integral part of our geometrical representation of two-elemental logic which is clearly evident at a first glance (see Fig.3). These squares are found to be dual since the four vertices of the outer square ($a + b$, $a + b'$, $a' + b$, $a' + b'$) correspond dually to the four sides of the inner square (ab, ab', $a'b$, and $a'b'$), whilst the four vertices of the inner square ($\underline{a}$, $\underline{a'}$, $\underline{b}$, and $\underline{b'}$) correspond dually to the four sides of the outer square ($\underline{a}$, $\underline{a'}$, $\underline{b}$, and $\underline{b'}$). The ten logico-geometrical elements remaining will be strictly bound up with our dual squares, provided these are conceived as being "complete squares".

With this object in view, we must refer to the concepts of "a complete quadrangle" and "a complete quadrilateral" which play a most important rôle in the newer synthetical (or projective) geometry, and are straitly connected with the concept of harmonic structures.

A "complete quadrangle" is a figure determinate by four vertices, which form (in six combinations of two elements) six straight lines, six sides of a complete quadrangle. These six sides of a complete quadrangle form three pairs of opposite sides (i.e., not passing through the same vertex), intersecting at three points - at the so-called diagonal points of the quadrangle.

Dually, a "complete quadrilateral" is a figure determinate by four sides which form (in six combinations of two elements) six points, six vertices of a complete quadrilateral. These six vertices of a quadrilateral form three pairs of opposite vertices (i.e., not lying on the same side of the quadrilateral), joined by three straight lines - the so-called diagonal lines of the quadrilateral.

Let us now again glance at Fig.3 with the concepts of "a complete quadrangle" and "a complete quadrilateral" in mind: our fundamental squares will now be readily grasped as complete squares.

Thus, first of all, the inner square, as "a complete quadrangle", will possess:

4 determinant vertices ($\underline{a}$, $\underline{a'}$, $\underline{b}$, $\underline{b'}$);

6 sides formed by these vertices, viz: four sides of the simple

~~square (ab, ab, ab, ab) and two diagonals~~

square $(ab, ab', a'b, a'b')$ and two diagonals[1]: $aa' = 0_{aa'}$ and $bb' =$ $= 0_{bb'}$; and

3 diagonal points, formed by the intersection (union) of three pairs of opposite sides, thus, the points: $ab + a'b$, $a'b + ab'$, and $aa' +$ $+ bb'$ $(= 0_{aa'} + 0_{bb'} = 0)$.

Dually, the outer square, as "a complete quadrilateral" will possess:

4 determinant sides $(\underline{a}, \underline{a'}, \underline{b}, \underline{b'})$;

6 vertices formed by these sides, namely four vertices of the simple square $(a + b, a + b', a' + b, a' + b')$, and the two diagonal points[2]: $a + a' = 1_{a+a'}$ and $b + b' = 1_{b+b'}$, and

3 diagonal lines, formed by the joining of three pairs of opposite vertices, (thus, the lines: $(a + b)(a' + b')$, $(a' + b)(a + b')$ and $(a + a')(b + b')$ $(= 1_{a+a'} \cdot 1_{b+b'} = 1)$.

In such wise, the inner square, as "a complete quadrangle" is a set of thirteen of its elements:

4 vertices, 6 sides and 3 diagonal points.

And dually: the outer square, as a "complete quadrilateral" is a set of thirteen elements dual to the preceding ones:

4 sides, 6 vertices and 3 diagonal lines.

We can now represent the above twenty-six elements of the categorial plane (cf. Table I, p.38) in the form of two dual series likewise in such fashion, that all the elements of the first series will be straight lines, whilst all the elements of the second series dual to them will be points. Using the terminology of "complete squares" these dual series - each of which contains thirteen elements - will appear as follows:

6 sides of inner square	4 sides of outer square
$0_{aa'}, 0_{bb'}, ab, ab', a'b, a'b'$;	a, a', b, b';

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[1] Should we in the case of the inner complete square take yet a third diagonal as the line joining its diagonal points: $ab + a'b'$ and $a'b + ab'$, we should receive a seventh side of the square - a straight line at infinity $(ab + a'b') \cdot (a'b + ab') = [0]$, one coinciding, it is true, with the third diagonal of the outer square: $(a + a')(b + b') = 1_{a+a'} \cdot 1_{b+b'} = 1$, but nevertheless of a different denomination to it (cf. footnote 1), p.41).

[2] Should we in the case of the outer complete square take yet a third diagonal point of intersection of its diagonals $(a + b)(a' + b')$ and $(a' + b)(a + b')$, we should receive a seventh vertex of such square, the origin of the co-ordinates $(a + b)(a' + b') + (a' + b)(a + b') = [1]$, which, though coinciding with the third diagonal point of the inner square $(= aa' + bb' = 0_{aa'} + 0_{bb'} = 0)$, is nevertheless of a different denomination to it (cf. footnote 1), p.41).

3 diagonals of outer square

$(a + b)(a' + b')$, $(a' + b)(a + b')$, 1

6 vertices of outer square

$1_{a+a'}$, $1_{b+b'}$, $a+b$, $a+b'$, $a'+b$, $a'+b'$;

4 vertices of inner square

a, a', b, b';

3 diagonal points of inner square

$ab + a'b'$, $a'b + ab'$, 0

This signifies: all the elements of the categorial logico-geometrical plane are found to be elements of our fundamental dual squares conceived as dual "complete squares"[1],or otherwise: a categorial logico-geometrical plane consists of two dual "complete squares".

We can also establish correspondence not only between the dual elements of the fundamental complete squares,but also between their elements of the same form,thus between points and points,and straight lines and straight lines. Such correspondence - known in projective geometry as homographic correspondence - can be represented as follows:

7 straight lines[2] of the inner square: $\lceil 0 \rfloor$, $0_{aa'}$, $0_{bb'}$, ab, ab', $a'b'$, $a'b$

7 straight lines of the outer square: 1, $(a'+b)(a+b')$, $(a+b)(a'+b')$, b, a, b', a'

7 points of the inner square: 0, $(a'b+ab')$, $(ab+a'b')$, b, a, b', a'

7 points of the outer square: $[1]$, $1_{a+a'}$, $1_{b+b'}$, $(a+b)$, $(a+b')$, $(a'+b')$, $(a'+b)$

Each of these elements corresponding to each other plays within the square,to which it belongs,the same rôle which the element corresponding to it plays in the other square. For instance,the opposite proper sides of the outer square <u>a</u> and <u>a'</u> correspond in the above table to the opposite proper sides of the inner square ab' and $a'b$,and so on. Thanks to the determination of this correspondence in the field of geometrical logic,we shall be able to pass from the relations between logico-geometrical elements of one square to the relations of the elements of the other square corresponding to them.

We shall now draw up a scheme of division of the elements of the categorial plane from the point of view of their denomination. We can distinguish uni-denominational elements (e.g.,a,a',aa',$a + a'$) and bi-denominational ones (e.g.,ab,$a + b$). After eliminating the origin of

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[1] Including as their elements diagonal points and diagonals.

[2] Including the 7th side $\lceil 0 \rfloor$ of the inner square and the 7th vertex $\lceil 1 \rfloor$ of the outer square (cf.footnote,p.46).

the co-ordinates ($0 = [1]$) and the straight line at infinity ($1 = [0]$), as occupying quite different status owing to their very complicated nature, we receive 12 simple and 12 complex elements.

U n i - d e n o m i n a t i o n a l   e l e m e n t s

points: $a, b, a', b', 1_{a+a'}, 1_{b+b'}$
straight lines: $a, b, a', b', 0_{aa'}, 0_{bb'}$

B i - d e n o m i n a t i o n a l   e l e m e n t s

points: $a + b, a' + b, a' + b', a + b', ab + a'b'$[1], $a'b + ab'$[2]
straight lines: $ab, a'b, a'b', ab', (a + b)(a' + b')(a' + b)(a + b')$.

The above scheme of division of the elements can be effected in conjunction with a division from the point of view of their finity and non-finity. Non-finite elements will be taken to be those which lie at infinity and their dualities (representing a system of ordinary and slanting axes of co-ordinates). Similarly, as above, we shall not for the moment examine the straight line at infinity and its duality, the origin of the co-ordinates. We thus have:

F i n i t e   e l e m e n t s.

U n i - d e n o m i n a t i o n a l   e l e m e n t s

points: a, b, a', b'
straight lines: a, b, a', b'

B i - d e n o m i n a t i o n a l   e l e m e n t s

$a + b, a' + b, a' + b', a + b'$
$ab, a'b, a'b', ab'$

N o n - f i n i t e   e l e m e n t s.

U n i - d e n o m i n a t i o n a l   e l e m e n t s

points: $1_{a+a'}, 1_{b+b'}$
straight lines: $0_{aa'}, 0_{bb'}$

B i - d e n o m i n a t i o n a l   e l e m e n t s[3]

$ab + a'b', a'b + ab'$
$(a + b)(a' + b'), (a' + b)(a + b')$

As will have been noted, we have here 16 finite and 8 non-finite elements in such wise that every non-finite element develops and yields two finite elements; e.g., the straight line $0_{aa'}$ - two points a and a'; point $ab + a'b'$ - two straight lines ab and $a'b'$, and so on. The non-fi-

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1) $ab + a'b' = (a' + b)(a + b')$.
2) $a'b + ab' = (a + b)(a' + b')$.
3) These elements could have been called: semi-non-finite.

nite elements also include the two supreme elements: the straight line at infinity and the origin of the co-ordinates, which unite the four non-finite points and the four non-finite straight lines (cf. pp. and , series 13a and 13b).

C H A P T E R V.

Sets of Four and of Six Elements of the Categorial Plane

To dual complex elements - the logical sum and the logical product- in mathematical logic correspond two dual complex operations: logical addition and multiplication (see p. 9). If, however, we have complex operations, we must likewise consider simple operations, such which will yield simple elements, such as a, b, or a', b'. Algebraic logic, as we know, deals only with one simple operation, viz. with negation. Yet side by side with negation and even before it, we should place a still more elementary operation, viz. the affirmation (position) which evokes the positive simple elements (a, b) only on the basis of which negation creates negative simple elements (a', b'). Thus, since geometrical logic leads to the discovery of two forms of simple elements (both positive and negative), the simple operation of affirmation and negation (thesis and antithesis) appears in dual form, absolutely analogously to the dual form in which the complex operations of addition and multiplication appear. If, however, we now inquire whether there is not such a correspondence for complex operations as for simple ones in the case of the correspondence between position and negation, the reply is an affirmative one, since we can of course speak not only of the synthesis a and b in a + b, but also of analysis - of the division of a + b into the simple elements a and b. Algebraic logic does not consider these analytical operations, if only for the simple reason that they actually find expression in the same formulae as synthetical operations, although with the difference that the formulae are examined in a different direction. None the less, these are operations diametrically contrary to synthetical ones and as such should be singled out. In such wise, the full table of logical operations, one likewise including simple operations in their dual form, appears as follows:

Logical Operations

- simple
  - positive (thetical)
    - fundamental
    - dual
  - negative[1] (antithetical)
    - fundamental
    - dual
- complex
  - (synthetical)
    - fundamental[2] (addition)
    - dual[2] (multiplication)
  - (analytical)
    - fundamental (converse of addition)
    - dual (converse of multiplication)

Thus, geometrico-architectonic logic distinguishes four simple operations (instead of the single simple operation accepted in algebraic logic - the operation of negation): (1) fundamental position (thesis); (2) position (thesis), dual as regards the fundamental one ; (3) fundamental negation (antithesis); (4) negation (antithesis), dual as regards the fundamental one. Such duality in the field of negative concepts already exists in traditional logic, which distinguishes the concepts "bad" and "not good"; the concept "bad" is the contrary (polar) one of the concept "good", whilst the concept "not good" is the negation proper of the concept "good". Such in fact, is the significance of the dual negative (or antithetical) elements in structural logic: one of them is the <u>pole</u> of the fundamental positive (thetical) element, and the other is its negation proper (here,[3] in the significance of privation, the lack of a positive element). The relation of the polar element to the negative one (in the strict sense of the term) is found to be one of duality, fully analogous to the relation which the fundamental positive element bears to the dual positive element. Hence, these four simple elements are the following:

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[1] In the broad sense of the term.

[2] It is a relative and arbitrary matter which of the two dual "complex" operations is accepted as the fundamental one.

[3] It may be mentioned that the negative element in the proper sense of the term (i.e. not a polar element as regards a given one) need not necessarily be always taken, however, as a lack - as the privation of something positive. It can, as a dual form of the polar element, merely represent the transfer of this element from the fundamental sphere to the dual one and can - as in the case with the polar element - represent the feature possessed and not the lacking one.

(1) the fundamental positive element;

(2) the element dual to (1);

(3) the element polar to (1);

(4) the element negative to (1);

whilst it should be noted that not only is the element (2) dual to (1) but also element (4) is dual to (3) being negative to (1).

We shall now examine these four elements on a logical plane (Fig.3)

Let us take the point a as the fundamental positive element. The straight line a will then represent the positive element dual to the point a; the point a' will correspondingly be the fundamental negative element (in the broader sense of the term), and the straight line a' will represent the negative element dual to the point a'. The point a' will be simultaneously a polar element, i.e. a pole as regards the point a, whilst the straight line a' will in relation to point a be its negation.[1] Moreover, we can transfer the relation of polarity from the sphere of punctual elements to equivalent dual elements, i.e. such as belong to the linear sphere, and thus state that not only the points a and a' (elements 1 and 3) constitute poles of each other, but also that the straight lines a and a' (elements 2 and 4, i.e. dual and negative as regards the fundamental positive element 1) are likewise polar in relation to each other. In such wise, not only the dual structure is transferred from the positive to the negative sphere, but also the polar structure passes from the fundamental sphere to the one which is dual as regards it.

The equivalent simple elements can be distinguished algebraically thus:

(1) the fundamental positive element: $a = a + 0$ (point a);

(2) the element dual to (1): $a = a.1$ (straight line a);

(3) the element polar to (1): $a' = a' + 0$ (point a');

(4) the element negative to (1): $a' = a'.1$ (straight line a').

With the passage of a given element to the dual one we see the application of the rule of passage from a given formula to the dual one (see p.9) just as with the passage from a given element to its negation de Morgan's rules (see p.21) are followed. Polar elements are not distinguished in algebraic logic from negative ones, and hence algebraic logic gives no rules for the passage from given elements to their poles.

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[1] The straight line a' is expressed by the equation: $a' = a'.1$; the point a by $a = a + 0$, and the point a' by $a' = a' + 0$. In accordance with de Morgan's formulae (see p.21), we have: $(a + 0)' = a'.1$, and this signifies that the negation of the point a is the straight line a', whilst the point a' will not be now the negation of the point a, but its pole.

However, as can be seen from the above list, in order to pass for instance from the element a + 0 to its pole, <u>a</u> has to be changed into its pole <u>a'</u> (and the straight line <u>a</u> into the straight line <u>a'</u>), the plus sign must be left uchanged, and instead of the straight line 0, we must take its pole which will also be the straight line 0. The pole of zero is therefore zero, and it passes into unity only under the action of negation and duality; just as the pole of unity is also unity and not zero.[1] Taking this more generally, it can now be stated: when passing from the sum (or product) of two or more elements to the polar element, the constituents of the sum (or the factors of the product) should be supplanted by the elements polar to them, without changing the symbol of operation and leaving the 0 or the 1. It is possible, however, to avoid this new direct passage from the given elements to the polar ones by resolving them in two passages already known to us, viz., passages from a given element to a dual one and then to one negative as regards this dual one (or, too, in the reverse direction). And actually, as can be seen from the above list, the element a' + 0, which is polar as regards the fundamental a + 0, is the negation of the element a.1, dual in relation to the fundamental element (or conversely: is dual as regards the element a'.1, which is the negation of the fundamental element a + 0). Similarly, the dual element can be defined with the help of the polarity and negativity of the fundamental element, and the negative element by means of duality and polarity.

The relations between the four simple elements have become clear owing to the fact that the simple elements have been represented as the limits of complex elements (e.g.; <u>a</u> as a + b, where b = 0). There can thus be no doubt that such sets of four elements can likewise be found in the sphere of complex elements. This can in fact be very easily demonstrated. Let us glance again at Fig.3, and take the complex element a + b as our point of issue. We can at once discover the four elements in question by passing along the diagonal of the larger square.

(1) the fundamental element: a + b (point);

(2) the element dual to (1): ab (straight line);

(3) the element polar to (1): a' + b' (point);

(4) the element negative to (1): a'b' (straight line).

We secure the analogous set of four elements when we take the ele-

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[1] This equivalence of the poles of the limitary elements we see also in arithmetic, where the arithmetical 0 is ± 0, and similarly, ∞ is ± ∞ . It can be demonstrated that zero is the pole of logical zero, and that unity is the pole of logical unity, by the formulae aa' = 0 and a + a' = 1, which on passage to polar elements, give on the left-hand sides <u>a'a</u> and a' + a, i.e. again 0 and 1.

ment a' + b as our point of issue - a complex element which consists of a positive and of a negative element (in the broader sense of the term). In such wise, all the elements of the categorial plane can be represented in the form of sets of four (the fundamental element, and the elements which are respectively dual, polar and negative to it). We then receive the following table of the topologic elements of the categorial plane (excluding the supreme elements: the point of origin of the co-ordinates and the straight line at infinity), consisting of six sets of four elements, the first three sets representing one-denominational elements and the latter three bi-denominational ones.

Fundamental Element	point a	point b	point $1_{a+a'}$
Dual ,,	straight line a	straight line b	straight line $0_{aa'}$
Polar ,,	point a'	point b'	point $1_{b+b'}$
Negative ,,	straight line a'	straight line b'	straight line $0_{bb'}$
Fundamental Element	point a + b	point a' + b	point a'b + ab'
Dual ,,	straight line ab	straight line a'b	straight line (a' + b)(a + b')
Polar ,,	point a' + b'	point a + b'	straight line (a + b)(a' + b')
Negative ,,	straight line a'b'	straight line ab'	point ab + a'b'

We already know that the element dual to unity is equivalent to the negative to 1; in the same way, the element dual to a'b + ab' is equivalent to the element negative to that element; the same is the case with the elements dual and negative to 0 and (a' + b)(a + b') [= ab + a'b']. Thus it is only geometrical logic which can demonstrate <u>ad oculos</u> that these equivalent algebraic elements are in spite of all different, that, for instance, the element dual to the point $1_{a+a'}$ appears in the form of the straight line $0_{aa'}$ (dual elements are determined by polarity with respect to a circle, see p.), whilst the element negative to the point $1_{a+a'}$ appears in the form of the straight line $0_{bb'}$.

The above-mentioned six sets of four elements can yield us four sets of six elements (triangular), two of which consist of one-denominational elements and the corresponding triangles will have their base on the main axes, whilst the other two consist of bi-denominational elements and the triangles rest on the slanting axes (see Fig.10).

The structures are the following:

(I) the axis $O_{aa'}$, point $\underline{a}$, point $\underline{a'}$, straight line $\underline{a}$, straight line $\underline{a'}$, point at infinity $1_{a+a'}$;

(II) the axis $O_{bb'}$, point $\underline{b}$, point $\underline{b'}$, straight line $\underline{b}$, straight line $\underline{b'}$, point at infinity $1_{b+b'}$;

(III) the axis $(a + b)(a' + b')$, point $a + b$, point $a' + b'$, straight line ab, straight line $a'b'$, point at infinity $ab + a'b'$;

(IV) the axis $(a' + b)(a + b')$, point $a' + b$, point $a + b'$, straight line $a'b$, straight line ab', point at infinity $a'b + ab'$.

C H A P T E R VI.

Harmonic Elements in Geometrical Logic

We now pass to an examination of a matter which will probably furnish the most eloquent evidence of the fertility of the geometrical method in logic. Thanks to the geometrization of the world of logic, we shall be able to perceive and determine such types of fundamental properties which by the method of pure logic might never have been discovered. A careful examination of the geometrical representation of the logical plane indicates the following structure in the four vertices of the inner square. For instance, the vertex b is the meeting place of the two straight lines ab and a'b, their bisector - the axis $O_{bb'}$, and the straight line b, perpendicular to this bisector. Similarly in the case of the vertices b', a and a'. Every person, acquainted with the elements of projective geometry will at once recognize in this structure the pencil of harmonic straight lines with the vertex at point b, and similar pencils with the vertices at points b', a and a'.[1] In such wise, geometrical intuition leads direct to the discovery of harmonic elements in the field of logic. Let us now examine this matter more closely, and analytically control the results yielded by intuition.

The four points A, B, C and D of a given straight line are called the harmonic points, if we have:

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[1] Attention is drawn to the pencil of harmonic straight lines with the vertex at the centre O of the axis of co-ordinates, a pencil of four straight lines: $(a + b)(a' + b')$, $(a' + b)(a + b')$, $O_{aa'}$, and $O_{bb'}$; it will be noted that the harmonic character of this group of four straight lines is evidently connected with the existence of the outer square, as a complete quadrilateral, and can be deduced from its existence. The outer square has three pairs of opposite vertices: $(a + b) - (a' + b')$, $(a + b') - (a' + b)$ and $1_{a+a'} - 1_{b+b'}$. We know from projective geometry that four straight lines belonging to one pencil form a harmonic group if one pair of opposite vertices of a complete quadrilateral lies on one of these straight lines, a second pair of such vertices on another straight line, and if the third and fourth straight lines include one of each of the elements forming the third pair of opposite vertices. These conditions are met by the pencil of straight lines with the vertex at the centre of the axis of co-ordinates, since the first pair of the opposite vertices of the outer square, the points $(a + b)$ and $(a' + b')$, lies on the straight lines $(a + b)(a' + b')$, the second pair, the points $(a + b')$ and $(a' + b)$, lies on the straight line $(a' + b)(a + b')$, whilst each of the straight lines $O_{aa'}$ and $O_{bb'}$ includes one of the elements of the third pair of these vertices, viz. the straight line $O_{aa'}$ includes the point $1_{b+b'}$, and the straight line $O_{bb'}$, the point $1_{a+a'}$.

$$\frac{AC}{BC} : \frac{AD}{BD} = -1, \text{or}$$

$$\frac{AC}{CB} = \frac{AD}{BD} \text{ .... } (\alpha)$$

We then likewise state that we have two pairs of harmonic conjugate points, the pair A,B and the pair C,D.

We denote AB by x, AC by a, AD by b. (Fig.7).

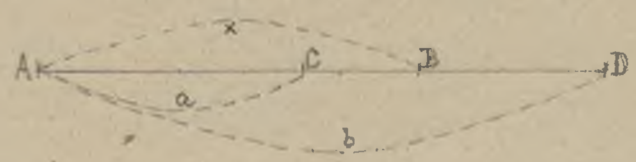

Fig. 7.

In accordance with the above definition of harmonic elements (α) the points A,B,C,and D will be harmonic points when we have:

$$\frac{a}{x - a} = \frac{b}{b - x} \text{ .... } (\beta)$$

We then also state that the segment AB = x has been harmonically divided by the points C and D, whilst the segment AB = x itself is called the harmonic mean of AC(a) and AD(b). The formula (β), representing continuous harmonic proportion, yields the following for the harmonic mean:

$$x = \frac{2ab}{a + b} \text{ .... } (\gamma)$$

We now denote the point of origin A by 0; the point C will then be denoted by a, the point B by x and the point D by b. On the basis of formulae (α) and (β), or (α) and (γ), we can state: the harmonic mean (x) is the fourth element harmonic to the three ones given: 0, a and b; or in other words: a harmonic element conjugate of 0 with respect to the conjugate harmonic pair a and b (cf. Fig.8).

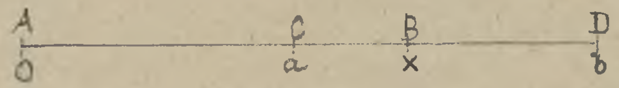

Fig. 8.

Let us now take as the point of origin A not 0 but ∞, and let us find the point which is harmonically conjugate with ∞ in respect of the fundamental pair of points a and b. Since the point A has now been shifted to ∞, we shall have AC = AD, and the equation (α) assumes the

form of CB = BD. If the point C be now designated as a, point D as b, and point B as X, we receive:

$$X - a = b - X \quad \ldots\ldots\ldots\ldots\ldots\ldots (\delta) \text{ or}$$

$$X = \frac{a + b}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\varepsilon)$$

The formula (δ) represents continuous arithmetical proportion and makes it possible to define X as the arithmetic mean of a and b (ε).

In such wise, not only the harmonic mean but also the arithmetic mean have been taken as the fourth point harmonic to the three given ones: the harmonic mean for the points a and b is the point x harmonically conjugate with 0 in respect of the points a and b; the arithmetical mean for a and b is the point X harmonically conjugate with ∞ in respect of points a and b.

Having made these elementary remarks on the connection between the concept of sets of four harmonic elements and the concepts of the arithmetical and of the harmonical mean, we shall now by purely geometrical methods, demonstrate the existence of harmonic sets in logic. The method applied is based on the theory of poles and polars in relation to the circle (Desargues, Poncelet, Gergonne) which plays such an important rôle in the new synthetic (projective) geometry and is so straitly connected with the problems of duality.[1)]

If we take a plane on which there are a given circle and a point P lying outside the circle (cf. Fig.9), and if from this point we trace lines cutting the circle (PS_1, PS_2, PS_3, etc.), then, the chords (AB, CD, EF, etc.) of the circle will be harmonically divided by the given point (P) and their points inside the circle (S_1, S_2, S_3, etc.), lying on a certain common straight line (KL).

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1) Attention is drawn to the fact that the theory of poles and polars in relation to the circle is not the sole purely geometrical method of introducing harmonic elements into topologic. We have seen in the footnote on page 56 that we can likewise attain this object by the concept of complete quadrilaterals (or quadrangles) as constituting harmonic groups.

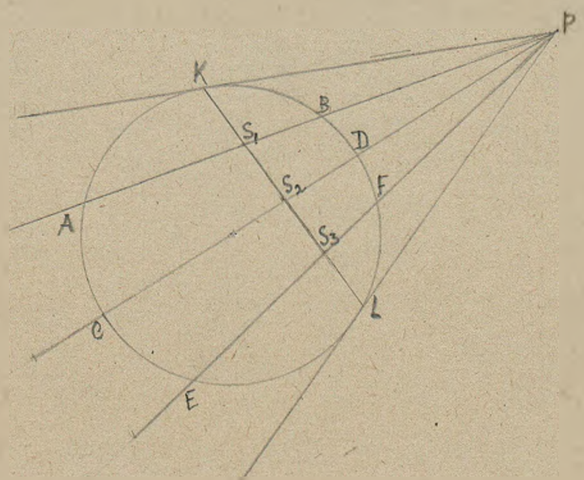

Fig. 9.

We obtain this straight line (KL) by tracing tangents to the circle from the given point P and then joining the points of tangency (K and L) with a straight line. In such wise we obtain sets of four harmonic points AS_1BP, CS_2DP, ES_3FP, and so on, in which the harmonically conjugated points will be points on the circle (e.g. A,B) and the given point P and the point S_1 on the line KL.

We shall call the straight line KL the polar of point P with respect to the given circle, and the point P the pole of the straight line KL. If the point P is shifted to infinity, its polar will be the diameter of the circle (at right angles to the line connecting the centre of the circle with the given point at infinity), and conversely: for every diameter of the circle the point at infinity (lying on a line perpendicular to such diameter) will be its pole. But if the point P is on the circle, its polar will be a tangent of the circle at that point, and conversely: the pole for every tangent of the circle will be its point of tangency.

We can now undertake to demonstrate the existence of harmonic elements in geometrical logic by applying the theory of poles and polars with respect to the circle, taking a harmonic group of logical elements to be a set of logical elements topologically corresponding to a set of geometrical elements forming a harmonic group. With this object in view, we draw two circles in our fundamental plane diagram, having their centre at the point of origin of the co-ordinates O: one of these circles around the smaller, and the other around the larger square, and the smaller circle will simultaneously be inscribed within the larger square ~~(see Fig. 10)~~[1]

(see Fig.10).[1]

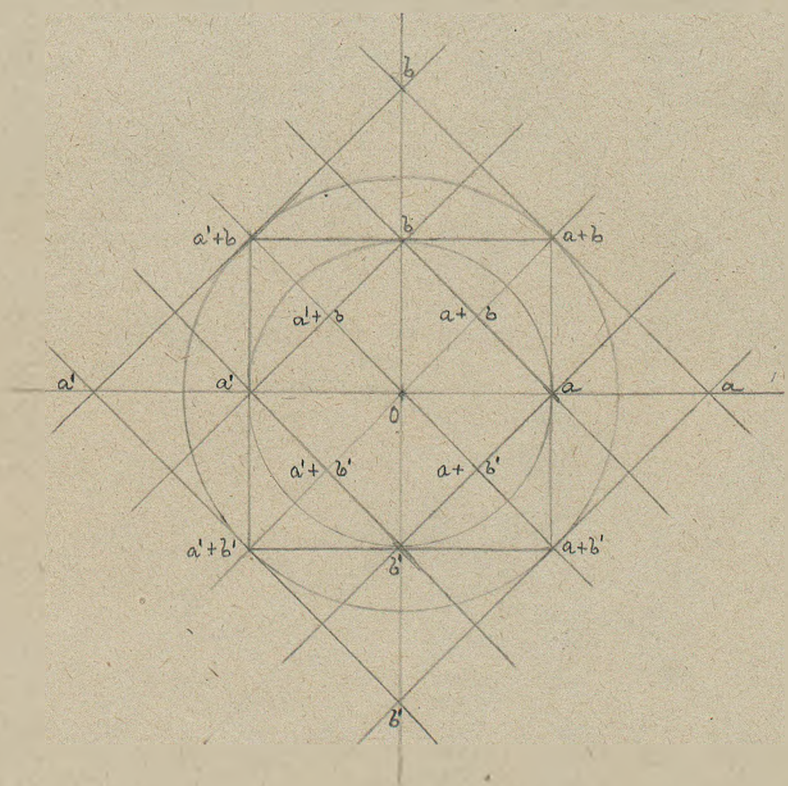

Fig. 10.

The next step is to trace from the points at infinity lines cutting these circles (the lines will of course be parallels). These points at infinity (in other words: directions on the plane) are so selected that the lines cutting the circles, their chords, are the sides and diagonals of the squares. Four points are found to be such points at infinity: (1) the point at infinity on the axis bb' ; (2) the point at infinity on the axis aa' ; (3) the point at infinity on the slanting axis (a + b). . (a' + b'), and (4) the point at infinity on the slanting axis (a' + b). . (a + b'). Three parallel straight lines a, a' and $0_{bb'}$, are traced from the first of these points at infinity ($1_{a+a'}$) - two chords of the larger circle and a diameter of the smaller one, and simultaneously two sides of the larger square and a diagonal of the smaller one. Similarly with the second point at infinity ($1_{b+b'}$); three parallel straight lines are drawn, b, b' and $0_{aa'}$ - two chords of the larger circle and a diameter of the smaller one, give the two remaining sides of the larger square and

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[1] In order not to complicate the diagram, the sides of the inner square (ab, ab', a'b, a'b') have not been marked, but the points of intersection of these sides with the diagonals of the larger square have been indicated, that is to say, points a + b, a + b', a' + b, a' + b', equivalents of the four vertices of the larger square; e.g., the intersection of ab with the diagonal (a + b).(a' + b') = a'b + a'b' yields the point ab + a'b + + ab' or the point a + b.

the second diagonal of the smaller one. The third of the points at infinity $a'b + ab'$, or $(a + b)(a' + b')$, gives us the three parallel straight lines $a'b$, ab' and $(a + b)(a' + b')$, namely two chords of the smaller circle and a diameter of the larger one, which are simultaneously two sides of the smaller square and a diagonal of the larger one. Similarly, the fourth point at infinity $ab + a'b'$, or $(a' + b)(a + b)$, gives us as two chords of the smaller circle, ab and ab' and as a diameter of the larger one, $(a' + b)(a + b')$, the two remaining sides of the smaller square and the other diagonal of the larger one.

Now, the theorems of polarity with respect to the circle can be easily applied to the twelve straight lines of our diagram, and each of these lines is then found to be the basis of a set of four harmonic points.

Let us take, for instance, the straight line $\underline{a}$, traced from the point $1_{a+a'}$, at infinity, as a pole with respect to the larger circle. The polar of the point $1_{a+a'}$, as a point at infinity will be, of course, the diameter perpendicular to the direction bb', or, what comes to the same, to the direction $\underline{a}$, i.e. the axis $O_{aa'}$. Thus we receive on the straight line $\underline{a}$, a harmonic set of four elements consisting of two points on the larger circle in which the secant cuts this circle as also of two other points harmonically conjugate: the pole $1_{a+a'}$ and point $\underline{a}$, in which the polar of point $1_{a+a'}$, i.e., the axis $O_{aa'}$, intersects our straight line $\underline{a}$. To the point P in Fig.9 corresponds here the point $1_{a+a'}$ at infinity, to its polar KI the polar of point $1_{a+a'}$, i.e., the axis $O_{aa'}$, and to the harmonic set of four points AS, BP the harmonic set of points $a + b'$, $\underline{a}$, $a + b$, and $1_{a+a'}$. Similarly with each of the other straight lines, as the basis of harmonic series.

Thus we receive the following twelve harmonic sets of four elements[1] which are given in spatial order so that the conjugate elements of one pair are divided by the conjugate elements of the second pair.

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[1] Attention is drawn to the fact that all the four elements of such harmonic sets are of the same form (here, points) and that therefore the sum of two elements will be an element of the same form as the components of such sum.

(1a)	$1_{a+a'}$,	$a + b$,	a,	$a + b'$	(basis a)
(2a)	$1_{a+a'}$,	b,	0,	b'	(basis $0_{bb'}$)
(3a)	$1_{a+a'}$,	$a' + b$,	a',	$a' + b'$	(basis a')
(4a)	$1_{b+b'}$,	$a + b$,	b,	$a' + b$	(basis b)
(5a)	$1_{b+b'}$,	a,	0,	a'	(basis $0_{aa'}$) 2)
(6a)	$1_{b+b'}$,	$a + b'$,	b',	$a' + b'$	(basis b')
(7a)	$(a + b)(a' + b')$	a,	$a + b'$,	b'	(basis ab')
(8a)	$(a + b)(a' + b')$,	$a + b$,	$[1]$ 1)	$a' + b'$	(basis $(a + b)(a'+b')$)
(9a)	$(a + b)(a' + b')$,	b,	$a' + b$,	a'	(basis $a'b$)
(10a)	$(a' + b)(a + b')$,	a,	$a + b$,	b	(basis ab)
(11a)	$(a' + b)(a + b')$,	$a + b$,	$[1]$ 1),	$a' + b$	(basis $(a + b')(a'+b)$)
(12a)	$(a' + b)(a + b')$,	b',	$a' + b'$,	a'	(basis $a'b'$)

It is necessary,also,to add the limitary harmonic series,the thirteenth on the straight line at infinity.

(13a) 3) $1_{a+a'}$, $(a + b)(a' + b')$, $1_{b+b'}$, $(a + b')(a' + b)$

If we now recall that algebraic expressions signify here not only geometrical elements (straight lines and points) but also concepts,the existence of harmonic sets of four elements in geometrical logic is thereby demonstrated.

Corresponding to the above twelve sets of logico-geometrical elements (and the thirteenth,limitary one),we can now easily receive the further twelve sets (and the corresponding thirteenth one),the elements of which will not be points but straight lines,whilst the linear basis will now be supplanted by a point-vertex. There will be harmonic pencils yielded by the above punctual sets by changing every point by its polar with respect to the circle,and the basis of a series of points by its pole. We are entitled to do this in accordance with the theorem of projective geometry which states that if some point lies on a straight line

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1) $[1]$ represents the point of origin of the co-ordinates. Since,for the slanting axis the point of origin of the co-ordinates,at which they intersect is expressed,as we know from the footnote 1) on page 41,not as 0 but as 1,as $(a + b)(a' + b') + (a' + b)(a + b') = 1$.

2) We can now impart a deeper sense to the definition of the negative element by means of the positive element as also 0 and 1 (cf. p.16),for we see that the set of four elements which concerns us here represents a harmonic set,so that the negative element $\underline{a'}$ can be defined as an element conjugate harmonically with the positive element $\underline{a}$ in the harmonic group in which the second pair of elements is 0 and 1.

3) The harmonic character of this group of points can be most simply deduced from the properties of the inner square (a complete quadrangle) which constitutes a harmonic group (see remarks on the dual case mentioned in the footnote on p.56).

its polar will pass through the pole of such straight line, so that to the four points of a given straight line will correspond the four straight lines passing through the pole of this straight line, and if these points form a harmonic series, then the corresponding straight lines (their polars) will likewise form a harmonic set (or harmonic pencil). In accordance with the foregoing, we receive from the set of four harmonic points (1a): $1_{a+a'}$, $a + b$, a, $a + b'$ with their straight-line basis a, the pencil of harmonic straight lines: $0_{aa'}$, ab, a, ab' with their vertex at point a, since, as we already know, the straight line a as a tangent of the circle, corresponds to the pole at the point of tangency a, and conversely; further the diameter $0_{aa'}$, as a polar corresponds to the point $1_{a+a'}$; and the polar ab corresponds to the point a + b as a straight line joining the points of tangency of the tangents traced to the inner circle from the point a + b; for the same reason the straight line ab' corresponds to the point a + b'.

It follows from the above that the relation between the pole and the polar is nothing else but the dual relation, i.e., that the pole and its polar represent dual elements.

Thus, we receive dually from the above sets of harmonic points the following pencils:

(1b) $0_{aa'}$,	ab,	a	ab'	(vertex a)
(2b) $0_{aa'}$,	b,	1,	b'	(vertex $1_{b+b'}$)
(3b) $0_{aa'}$,	a'b,	a',	a'b'	(vertex a')
(4b) $0_{bb'}$,	ab,	b,	a'b	(vertex b)
(5b) $0_{bb'}$,	a,	1,	a'	(vertex $1_{a+a'}$)
(6b) $0_{bb'}$,	ab',	b',	a'b'	(vertex b')
(7b) (a' + b)(a + b'),	a,	ab',	b'	(vertex a + b')
(8b) (a' + b)(a + b'),	ab,	[0],[1]	a'b'	(vertex (a' + b)(a+b'))
(9b) (a' + b)(a + b'),	b,	a'b	a'	(vertex a' + b)
(10b) (a + b)(a' + b'),	a,	ab,	b	(vertex a + b)
(11b) (a + b)(a' + b'),	ab ,	[0],[1]	a'b	(vertex (a + b)(a'+b'))
(12b) (a + b)(a' + b'),	b ,	a'b',	a'	(vertex a' + b')

To the foregoing we must add the thirteenth harmonic pencil, the limitary one having the vertex at the point of origin of the co-ordinates:

(13b) $0_{aa'}$, (a' + b)(a + b'), $0_{bb'}$, (a + b)(a' + b')

In such wise, the categorial logical plane is found to be the domain of harmonic elements: each of its twenty-six elements is the centre

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[1] [0] here represents a straight at infinity. Since, for slanting axes this line is found to be the product of (a + b)(a' + b') by (a' + b) (a + b').

(base or vertex) of a set of four harmonic elements and,moreover,every set of four of its elements having the same base or vertex is a harmonic set.

We learn,too,that each of these harmonic sets of four elements contains at least as one of its elements the limitary element,i.e.,the one situated at infinity (4 points and a straight line at infinity) or dual to it (4 axes and the point of origin of the co-ordinates),whilst the other element,conjugate with the first,represents the logical product or logical sum of the remaining two elements. Moreover,this product or this sum likewise express the product or sum of the first pair of conjugate elements and represent equivalently the basis or the vertex of the set of four elements. In the first group (in twelve harmonic punctual sets)[1] this basic element (straight line) signifies the same product of the elements of pairs harmonically conjugated; in the second group (in twelve harmonic linear groups)[1] the vertex element (point) signifies the same sum of elements belonging to the pairs harmonically conjugate.

Since we have,by purely geometrical means,ascertained the existence of harmonic sets in logic,it is now easy to find the relation of these logical sets to certain arithmetical ones. It will suffice to consider the above-mentioned properties (pp.57 and 58) of the arithmetical and of the harmonic mean,as elements of arithmetical harmonic sets of four elements and to compare them with the corresponding properties of logical harmonic sets of four elements. The comparison can be effected as follows:

The arithmetical mean or the harmonic mean of two given numbers represents one of the elements of a harmonic arithmetical set of four; conjugate with it will be the limitary element (∞,0),and the two remaining ones will be the two given numbers.

The logical product or the logical sum of two given logical elements represents one of the elements of a harmonic logical set of four; conjugate with such product (or sum)will be the limitary element 1,0 or $(a+b)(a'+b'),(a'+b)(a+b')$,and the two remaining ones will be the two given logical elements.

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[1] The thirteenth pair,the absolute one,exclusively composed of limitary elements,possesses a specific quality which will not be more closely examined here - one which arises from the property of the point which is the origin of the co-ordinates and of the line at infinity; namely: each of these elements in dependence on the manner in which it is determined (by the horizontal or by the slanting axes) represents either 0,or 1. Here, 0 becomes equivalent to 1.

We are at once struck by the analogy between the arithmetical or the harmonic mean and the logical function of the product or sum; this analogy will be found to be a real one, since in very fact, by introducing a correspondence between the logical negation and the arithmetical reciprocal, it is possible to prove that the relation of the logical product to the logical sum is identical with the relation of the arithmetical to the harmonic mean.

For, we know the definition of the harmonic mean of two elements as the reciprocal of the arithmetical mean taken for the reciprocal of these elements.

Let these elements be a and b. We therefore have:

Arithmetical Mean: $\dfrac{a + b}{2}$

Arith. Mean of their reciprocal: $\dfrac{\frac{1}{a} + \frac{1}{b}}{2}$

Harmonic Mean: $\dfrac{2}{\frac{1}{a} + \frac{1}{b}} = \dfrac{2ab}{a + b}$

Identical relations exist between the logical product and sum. For we can define the logical sum of two elements as the negation of the logical product taken for the negation of these elements (de Morgan's formula 9^{b}, see p.21).

Hence:

Logical product of the elements: $a \times b$
Logical product of their negation: $a' \times b'$
Logical sum: $(a' \times b')' = a + b$.

We see therefore, that the harmonic mean arises out of the arithmetical mean in the same way as the logical sum arises out of the logical product, namely by the substitution of the elements by their reciprocal (negation), followed by the reciprocal (negation) of the result yielded by this operation. We have now demonstrated the precise analogy between the logical sum and product on the one hand, and the harmonic and the arithmetical mean on the other.

C H A P T E R VII.

The Dichotomical and Tetrachotomical Harmonic Division of Concepts in Geometrical Logic

The chapter on the division of concepts is most closely connected with the science of harmonic elements in logic. The fundamental formulae expressing the connexion between a concept subjected to division and the products of such division are the dual formulae of dichotomy:

$$a = (a + b)(a + b') \quad \ldots\ldots\ldots\ldots\ldots\ldots (5^a)$$

$$a = ab + ab' \quad \ldots\ldots\ldots\ldots\ldots\ldots (5^b)$$

We already know that the genus a and its species (a + b) and (a + b') together with the element 1 present a set of four harmonic elements, and that the same holds good in the case of the elements of formula 5^b with the element 0.

We shall examine these formulae, particularly 5^b, before proceeding farther. It expresses the logical division of the element a into two elements smaller than a, viz. ab and ab'. The query now arises: What is the relation of the elements, appearing in this division, to the harmonic set 0,ab,a,ab' (or the one equivalent to it: 0,ab',a,ab)?[1] Primarily, it is necessary to elucidate the rôle of the element 0, not revealed in formula 5^b, but none the less fully implied in it, since a = ab + ab' is equivalent to a + 0 = ab + ab'.

This matter is quite analogous to the harmonic quantitative division, when the segment AB (cf. Fig.8 and 11) is harmonically divided by the points C and D, whilst the point of origin A is taken as 0 (arithmetical); the elements ab and ab' would then correspond - in accord with the exact analogy between the logical product and the arithmetical mean - to the arithmetical elements:

$$\frac{a + b}{2} \text{ and } \frac{a + \frac{1}{b}}{2} \left(= \frac{ab + 1}{2b} \right)$$

If, for instance, a = 1, b = 2, we would then have:

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[1] The harmonic set 0,ab',a,ab is equivalent to the set 0,ab,a,ab' (set 1b, on p.63), as can be easily seen in Fig.3. For we can just as easily, in order to secure the harmonic set of four, revolve the axis $0_{aa'}$ through ab as through ab'.

arithmetical set:
strictly analogous
to the logical set:

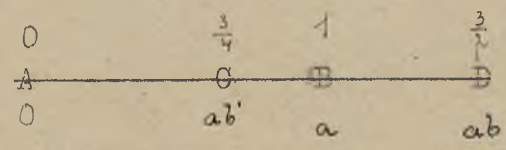

Fig. 11.

Thus, similarly as the arithmetical distance 0 - 1 has been harmonically divided by the fractions $\frac{3}{4}$ and $\frac{3}{2}$ [i.e., divided internally and externally in the same ratio $\left(\frac{3}{4} - 0\right) : \left(1 - \frac{3}{4}\right) = \left(\frac{3}{2} - 0\right) : \left(\frac{3}{2} - 1\right)$] in the same way the logical distance 0 - a has been harmonically divided by the logical elements ab' and ab. This signifies too, that instead of stating, formula 5^b expresses the harmonic division of the logical element a into the elements ab' and ab, we can affirm that formula 5^b expresses the harmonic division of the logical distance 0 - a by the above elements. We thus see that the rôle of the logical element 0 is the same as that of the arithmetical 0 in the analogous arithmetical situation: it is the point of issue of the division of a certain distance, which, in view of the modulus character of such point of issue can be equivalently expressed by its limitary point (the arithmetical distance 0 - 1 by the number-point 1, and the logical distance 0 - a by the concept-point a).

As we have just seen, the ordinary logical division of the concept a is in reality an harmonic division of the qualitative distance (difference), which separates this concept from 0. This distance 0 - a should be divided into two logical distances, the logical sum of which will be equal (equivalent) to the given logical distance (just as with the harmonic division of the arithmetical distance having its point of issue at 0, we divide it into two distances, the harmonic mean of which equals the given distance).

Thus, in accordance with the theory of harmonic elements in logic, the dichotomic formula:

$$a = ab + ab',$$

which represents the logical sum of two logical elements, simultaneously represents the logical sum of two logical distances (from 0 to ab, and from 0 to ab'), whilst the distance a, being a sum, is harmonically conjugate with 0 in respect of the elements of this division; so that the equation $a + 0 = ab + ab' = a$ here expresses the fundamental property of harmonic sets (cf. p.64): equivalence of the sums [or products] of elements conjugate in pairs.

Let us take an example in order to illustrate this theory of logical dichotomy.

Let a be here the comprehension "man", b "a good being", b' "a being who is not good",[1] so that ab means "either a man or a good being", and ab', signifies "either a man or a being who is not good". To divide, in the proper sense, concept a in view of b, b' is nothing else than to find two such concepts composed of a and b, or a and b', which together (+) fill up the distance separating the concept a ("man") from the zero comprehension (i.e., the object in general).

Now, let the first of these comprehensions smaller than a be the comprehension ab, the comprehension "either a man or a good being"; this is a comprehension which is already larger than O (the object in general), but is still smaller than a, less determinate than a; it is not yet the comprehension "man" but "either a man or a good being". In order now to ascertain the comprehension of a, in order to pass along the whole distance from O to a, to fill up all of this difference, it is necessary to make a second step, viz., from O to ab', or in other words, to create the comprehension "either a man or a being who is not good". For, if the comprehension of this ab' is added to the former ab, we receive a comprehension which has its determination = a, and will fill with these combined steps the difference which separates, with regard to determination, the comprehension a from the comprehension O.

This harmonic division of the comprehension a is shown in Fig.3 in such wise that the distance of comprehension a from the comprehension of O is represented here in the form of an angle between the straight line a and the axis $O_{aa'}$, and this distance is divided by the straight lines (comprehensions) ab and ab', which divide this angle internally and externally in accord with the rule of harmonic groups.

We have so far dealt only with the second of the two dichotomic formulae (5^b). We began with it since it expresses division in the proper sense, when the members of the division are smaller than the divided element ($ab < a, ab' < a$). The dual dichotomic formula 5^a does not express such a division: its elements are larger, richer in comprehension, and more determinate than the concept subject to division ($a + b > a, a+b' > a$). Strictly speaking, we have here not a division proper but determination (specification) or determining division as opposed to "dividing" division proper.

There is strict dual correspondence between these two divisions:

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[1] Cf. footnote) on p. 7 .

Dividing Dichotomy	Determining Dichotomy
$(a = ab + ab')$	$a = (a + b)(a + b')$
Element subject to division	Element subject to determination
(totum dividendum: a)	(determinandum: a)
Dividing factor	Determining factor
(dividens: b, b')	(determinans: b, b')
Member-products of division	Member-products of determination
(membra divisa: ab, ab')	[membra determinata: $(a + b), (a + b')$]

Now, all that we have so far stated as regards dividing dichotomy, can be dually transferred to determining dichotomy, but with one difference: that the comprehension a will here refer to the limitary comprehension 1. Here, in the case of determining dichotomy the point of issue is not logical 0, but logical 1, whilst the dual "distances" are expressed by the product of a given comprehension and 1, in reality indicating what the given element has in common with 1. The comprehension a, common to a and 1, is attained here by means of two logical steps, issuing from 1. The first step does not reach a, but it diminishes the comprehension 1 to $a + b$, the second step, on the other hand, diminishes it to $a + b'$; taking what these two steps have in common, we attain a diminution of the comprehension 1 to a. In this sense, the "distance" $a.1$ is harmonically divided by the comprehension-distances $(a + b)$ and $(a + b')$, of which it is "composed" by multiplication, and the equation $a.1 = (a + b)(a + b') = a$ expresses the fundamental property of harmonic sets: equivalence of the products [or sums] of elements in conjugate pairs.

This harmonic determination (specification) of genus comprehension into two specific comprehensions is depicted in Fig.3 in such wise, that the harmonic group is given on the straight line a, consisting of the points $a + b'$, a, $a + b$ and of the point at infinity 1 ($= a + a'$). The distance $a - 1$ ($= a.1$) is here divided by the points $a + b$ and $a + b'$, dividing it internally and externally in accord with the law of harmonic groups.

x x

x

We can now examine the harmonic logical divisions from another point of view, viz., by taking the elements determining the logical distance already as the products of division. Let us now examine, from this standpoint, the formulae which express the harmonic nature of logical

divisions:

$$a = a.1 = (a + b)(a + b') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5^{a'})$$

$$a = a + 0 = ab + ab' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5^{b'})$$

we are at once struck by the fact that both sides of the equation $a.1 = (a + b)(a + b')$ have the same bi-member structure and that the same can be stated as regards the sides of the dual equation: $a + 0 = ab+ab'$. We shall secure even stronger conviction as regards the actual affinity in the nature of both sides of our equations, if the expression $a.1$ (the distance $a - 1$) and $a + 0$ (the distance $a - 0$) are represented as follows:

$$a.1 = (a + 0)(a + 1)$$

$$a + 0 = a.1 + a.0$$

We shall then at once perceive that $a.1$ is nothing else but the limitary case $(a + b)(a + b')$ for $b = 0$, whilst $a + 0$ represents the limitary case $ab + ab'$ for $b = 1$. If then the elements $(a + b)$ and $(a + b')$ are already products of division, there is nothing to hinder us from likewise considering the elements <u>a</u> and 1 as the products of dichotomic division. The same applies dually to the elements <u>a</u> and 0 of the dual formula. But in such case the questions arise: Which element here undergoes this two-fold dichotomy? What do we divide tetrachotomically in such case, which is the determinandum or the dividendum?

Replies are yielded by recourse to formulae $5^{a'}$ and $5^{b'}$. Each of these formulae, apart from the four elements examined by us above and comprising the harmonic groups of four elements, also contains a fifth element, viz. the element <u>a</u>, still simple, undeveloped, equivalent to the product $a.1$ or $a + 0$ and to the product $(a + b)(a + b')$ or $ab + ab'$. It is this simple <u>a</u> which develops tetrachotomically, is twice divided dichotomically, and which is the determinandum or dividendum sought by us. We can easily ascertain the geometrical nature of this <u>a</u>: the simple <u>a</u> of the formula $5^{a'}$ is the straight line <u>a</u> which acts as the basis of the harmonic group of points: $1, a + b, a, a + b'$; whilst the simple <u>a</u> of the formula $5^{b'}$ is the point <u>a</u>, the vertex of the harmonic pencil of straight lines: $0, ab, a, ab'$, and as such this simple <u>a</u> will be equivalent to the two products, or to the two sums of the elements harmonically conjugate (cf. p. 64). What, however, does the logical simplicity of this element <u>a</u> signify? It denotes its still undeterminate, undivided nature; this actually undetermined <u>a</u> - a straight line - becomes twice determined[1]

[1] By the intersection with the lines 0 and 1 (a straight line at infinity) and the lines b and b'.

as a.1 and as $(a + b)(a + b')$, i.e. once as a straight line passing through the point a and the point 1, and the second time as a straight line passing through the points $(a + b)$ and $(a + b')$; whilst the actually indeterminate, simple point a becomes twice determined[1] as $a + 0$ and $ab + ab'$, that is, once as the point of intersection of the straight lines a and 0, and the second time as the point of intersection of the straight lines ab and ab'. In such wise, on the straight line a its four point-determinations appear, whilst the point a is cut by four line-components, products of the tetrachotomical division or development of the simple element a.

From the dual formulae for dichotomy, $5^{a'}$ and $5^{b'}$, there is no difficulty in receiving the formulae for dual harmonic tetrachotomical division (development), combining in itself the two dichotomical divisions in the following form:

$$a = (a + b).(a + b').a.1 \quad \ldots\ldots\ldots\ldots (V^{a})$$

$$a = ab + ab' + a + 0 \quad \ldots\ldots\ldots\ldots (V^{b})$$

or, expressed in fuller form:

$$a = (a + b)(a + b')(a + 0)(a + 1) \quad \ldots\ldots\ldots\ldots (V^{a'})$$

$$a = ab + ab' + a.1 + a.0 \quad \ldots\ldots\ldots\ldots (V^{b'})$$

And finally in most complete form:

$$a = (a + b)(a + b')(a + bb')(a + b + b') \quad \ldots\ldots\ldots\ldots (V^{a''})$$

$$a = ab + ab' + a.(b + b') + a.bb' \quad \ldots\ldots\ldots\ldots (V^{b''})$$

Let a be here the comprehension "man", b the comprehension "good", and b' the polar comprehension "bad". Then formula $V^{a''}$ will signify: a man is either "a good man" or "a bad man", or "a good or bad man", or "a good and bad man". The last two possibilities require fuller explanation.

Thus, first of all, what is "a good or bad man"? It is a man who has a moral character which is univocally undetermined ("or"), changeable, vacillating, hesitating on one side or on another, appearing now in one rôle and then in another. On the other hand, a man whose character integrates within itself the contrary determinations of "good" and "bad" ("a good and bad man") in his moral supra-determination, signifying neutralization of contrary moral tendencies, is at once led out beyond the sphere of morality proper; he is a man situate beyond "good" and "bad", on the frontier where the neutralized contrary moral tendencies now exercise no influence. In such wise, the logical tetrachotomy of the concept "man", in view of his moral character is represented as follows: a man

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[1] By connexion with the points 1 and 0 and the points b and b'.

is either good, or bad, or vacillating between good and bad, or is beyond good and bad - and these four species as we already know represent a harmonic logical group.

All that we have so far stated about the division of the element a can be mutatis mutandis applied to the division of the element 0 or of its duality - the element 1.

Here we shall have:

$$O_{aa'} = 0.1 = aa' \text{ (harmonic dichotomic division)}$$
$$O_{aa'} = a.a'.0.1 \text{ (harmonic tetrachotomic division)} \qquad (A)$$

and dually:

$$1_{a+a'} = 1 + 0 = a + a' \text{ (harmonic dichotomic division)}$$
$$1_{a+a'} = a + a' + 1 + 0 \text{ (harmonic tetrachotomic division)} \qquad (B)$$

It now, following the example of the twofold dichotomy just mentioned, we should desire to develop our uni-dimensional harmonic tetrachotomy by the tetrachotomic division of each of its elements, we should enter upon a plane and receive the following harmonic sets of four elements, members of the division of the linear elements (a,a',0,1) (see Fig.3):

$$a = (a + b).(a + b').a.1_{a+a'}$$
$$a' = (a' + b).(a' + b').a'.1_{a+a'}$$
$$O_{bb'} = b.b'.0.1_{a+a'}$$
$$1 = (a + b)(a' + b').(a + b').(a' + b).1_{b+b'}.1_{a+a'} \quad ^{1)}$$

From which, substituing in (A), we receive the following thirteen-element development of 0.(reduced from the 16-element = 4^2):

$$O_{aa'} = (a + b).(a + b').a.(a' + b).(a' + b').a'.b.b'.0.$$
$$.[(a + b)(a' + b')].[(a + b')(a' + b)].1_{b+b'}.1_{a+a'}$$

And dually, by substituting in (B) the fourfold harmonic divisions of the points: a,a',1 and 0, we receive the following 13-element development of 1:

$$1_{a+a'} = ab + ab' + a + a'b + a'b' + a' + b + b' + 1 + [ab + a'b'] +$$
$$+ [ab' + a'b] + O_{bb'} + O_{aa'}.$$

As we see, all the elements - to the number of twenty-six - of the categorial plane can be secured by two-fold harmonic tetrachotomy of 0 and 1: the development of 0 as an axis, gives us thirteen points of this plane, and the dual development of 1, as a point at infinity, gives us its thirteen straight lines.

We see here the fundamental difference between dichotomical and

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1) Cf. the set of four elements 13a on p. 62; in the present chapter, the conjugate elements of the harmonic sets of four written one after another as belonging to the same dichotomy.

tetrachotomical division: whilst the twofold dichotomy of 0 and of 1 gives us only $2^2 + 2^2 = 8$ elements of the categorial plane, the twofold tetrachotomy of 0 and of 1 exhausts the totality of this plane ($4^2 + 4^2 = 32$, reduced to 26 as a result of the four-fold repetition of the element $1_{a+a'}$ and $0_{aa'}$).[1)]

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1) The question arises: How can already the dichotomic division of a concept (e.g. the concept of "man") be considered as complete? This is justified only then when its negative member (e.g. "a not-good man") signifies the multitude of men in whose nature "goodness" does not enter, and not the multitude of men in whose nature "not-goodness" (badness) enters. In the latter case, dichotomic division of course will not be exhaustive: between people who are "always good" and "always not good" ("always bad") there will be intermediate groups, just as between two contrary propositions A and E.

CHAPTER VIII.

Specification of Mathematical Pan-Logic

It was pointed out in the footnote on p.14 that the bi-elemental logical space, as seen in Fig.3, is a representation only of one specification of mathematical pan-logic, viz., of its most developed form. We now return to the consideration of this so important question. It will be seen that varied types of logic may exist, and that the factor responsible for the differentiation of the unity of general logic is to be found, in the first place, in relations existing between the simple elements $\underline{a}$, $\underline{a'}$, $\underline{b}$, $\underline{b'}$ (if we limit ourselves to two dimensions).

In speaking of the relations of the elements $\underline{a}$, $\underline{a'}$, $\underline{b}$, $\underline{b'}$ we must distinguish two different possible types of combinations between these elements, viz.: (1) relations between the antithetic elements (of the same denomination) $\underline{a} - \underline{a'}$ or $\underline{b} - \underline{b'}$, and (2) relations between elements of different denomination, such as between the element $\underline{a}$ and the element $\underline{b}$. These logical relations between two elements would be as follows: (1) either $a < a'$ (but $a' \nless a$), or $a' < a$ (but $a \nless a'$), or $a < a'$ and $a' < a$, i.e., $a = a'$, or, finally $a \nless a'$ and $a' \nless a$; similarly (2) either $a < b$ (but $b \nless a$), or $b < a$ (but $a \nless b$), or $a < b$ and $b < a$, i.e. $a = b$, or finally $a \nless b$ and $b \nless a$.

We shall first consider more closely the relations existing between antithetic elements, since the variety of these relations, as will appear, gives the basis for the natural differentiation of structural topologic. Of the four relations which may exist between the antithetic elements $\underline{a} - \underline{a'}$ (or $\underline{b} - \underline{b'}$), the case in which $a \nless a'$ and $a' \nless a$ is that which first of all deserves our attention. In this case, the relation between the antithetic elements is the least paradoxical and the least dialectical; none of the antithetic elements are contained in another, and they are entirely of the same order, and independent of each other. Such a relation would be presumed to exist between antithetic elements in every case, unless we possessed other indicative data of the existence of a different relations.

It may, however, be otherwise, and actually is so in the cases of the three remaining relations between antithetic elements. In these, one of the antithetic elements is contained in the other ($a < a'$, or $a' < a$), or they may be mutually contained within each other, and thus be equivalent

($a = a'$). These possibilities appear paradoxical to us, in view of the circumstance that the antithetic elements are negative (for the moment in the widest sense of the term) with respect to one another, and yet one of them is contained in the other, or the positive element is even to be equivalent to its negation. Does this not contravert the principle of excluded contradiction, and are such dialectical relations permissible in a logic which respects this principle? We know that algebraic logic admits the principle of excluded contradiction, and that it at the same time admits in its non-contradictory system the statement that $0 < 1$, 0 being the negation of 1. In exactly the same way as there is no contradiction in the proposition that logical unity contains within itself its negation, there is nothing conflicting in the statement that the element $\underline{a}$ contains its negation $\underline{a'}$, or vice versa.

This is undoubtedly the case, and yet the paradox is not thereby eliminated. The paradox which we are liable to see in dialectical relations of the type $a < a'$, vanishes entirely only when we realise that negation does not necessarily imply privation, and that the inclusion of its negation by an element does not in the least imply possession and non-possession simultaneously of the same characters or components (cf. footnote 3) on p.51).

It thus appears that exact logic can assume different forms, depending on the nature of the relations existing between the antithetic elements of a given domain. Two forms in particular distinguish themselves: the first, in which the simple antithetic elements are not inclusively inter-related, and the second in which the relation between antithetic elements is one of their uni-orbilateral inclusion. It should, however, be borne in mind that both these forms of exact logic are merely specifications of one and the same general, as yet undifferentiated logic, the formulae of which admit of various possibilities, in exactly the same way as the formulae of pan-geometry give Euclidean and non-Euclidean geometry in their specifications. And precisely in the same way as the systems of non-Euclidean geometry not only do not comprise logical contradictions, but even as systems of physical geometry possess real significances, so are the logical systems with antithetic elements connected by the relation of indubitable inclusion, and may possess a real significance. Further, in the same way as Euclidean geometry is not the only possible geometry, but is only a certain specification of general pan-geometry, so also is the most natural and common logic of antithetic, completely separate and independent elements (this is expressed geometrically by parallel straight lines), not the only possible logic, but only

a certain variety of pan-logic. This pan-logic is given in the formulae of geometro-algebraic logic, if these formulae are considered in the most general way. We shall not here further develop the details of this remarkable parallelism between the specifications of geometry and logic, but shall return to the abovementioned two fundamental varieties of mathematical logic, in order more closely to study them from the point of view of structure.

We shall commence with a few remarks on terminology. Logic, in which between elements a and a', or b and b' relations of inclusion exist, we shall, in accordance with philosophic tradition, term "specifically dialectic" logic, or, more simply, dialectic logic, whilst the logic dealing with mutually independent antithetic elements we shall term hemidialectic logic.[1)] We shall not term it ordinary logic because certain dialectic features partly enter into it, not, it is true, between the simple elements a and a', or b and b', but between the antithetic end-elements 0 and 1, between which a relation of inclusion exists ($0 < 1$). Apart from this it comprises, in relation to the elements 0 and 1, a synthesis of antithetic elements, of the type $a + a' = 1$, and $aa' = 0$; such synthesis are characteristic of dialectic logic. For these reasons this kind of mathematical logic must also be distinguished from ordinary logic (not recognising the elements 0 and 1), even though it does not include the extreme dialectic relation of the antithetic elements a and a'. The presence of certain dialectic features is therefore in fact suggested by the very term "hemidialectic logic".

Let us commence with specifically dialectic logic. Suppose the relation $a < a'$ holds in this; then $a + a' = a'$, and $aa' = a$ (cf. equation I, p.22). If the converse holds, i.e., $a' < a$, then $a + a' = a$, and $aa' = a'$. Finally, when $a = a'$, then $a + a' = a' = a = aa'$. As we see from this, additive and multiplicative combinations of antithetic elements do not yield new (unequivalent) elements, but lead to the same elements a or a'. The combination $a + a'$ does not here give unity such as it is usually regarded, i.e., a unity which is not equivalent to the simple elements a or a', and the same applies to the combination aa', with reference to 0. In such a logic, elements composed of antithetic elements reduce themselves, roughly speaking, to simple ones. This imparts a specific architectonic character to such a system of dialectic logic; the system is

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1) We shall see later that one of the two types of these logic dealing with antithetic elements not connected by the relation of inclusion represents a transition to dialectic logic; its antithetic elements, however, are not related by inclusion, but present common elements which are $\neq a$, $\neq a'$.

not freely developed: the elements composed of a and a' (or b and b') do not occupy separate places to those of simple elements. We have here a crowding, as it were, and an involution of elements, and the whole has the nature of a reduced, compressed system.

We shall now consider hemidialectic logic, in which the antithetic elements are not related by inclusion ($a \nless a'$, and $a' \nless a$). There can here be no question of a reduction such as was shown above, viz., a reduction of elements composed of antithetic elements to simple ones. For $a + a' = a$ (or a') only when $a' < a$ (or $a < a'$), and similarly for aa'; if then $a' \nless a$, and $a \nless a'$, such a reduction is impossible. In such a logic 1 and 0 can never be equivalent to the simple elements a or a'; they can never be congruent with them, and in this fundamental respect such a system of logic will possess far fuller and better developed architectonics than the preceding one. Whether these features are complete we do not yet know, since only this is certain, that neither 1 nor 0 can be reduced to the simple elements a, a' (or b, b'), but we do not know whether they might not be reduced to compound elements of the type $a + b$, with reference to unity, or of the type ab, in the case of zero. We may a priori understand that a system of logic will be completely developed architectonically only when 1 and 0 will be equivalent neither to the simple elements a, a', b, b', nor to elements compounded of simple ones (of different denominations). We shall, in order more closely to consider this matter, return to the discussion of the relations existing between the elements a or b (cf.p.74).

The simple elements a, a', b, b' can be developed in a normal, completely un-reduced dichotomical, and hence actually two-member form, only when relations of inclusion between the elements a and b (or b') are completely absent, viz., when:

$$b \nless a,\ b' \nless a,\ a \nless b,\ a \nless b'.$$

In such a case all dichotomic developments, equally of the element a as of the element a', of the element b as of b', and equally of the multiplicative type $[a = (a + b)(a + b'), \text{etc.}]$ and of the additive type $[a = ab + ab', \text{etc.}]$ will be complete, and entirely irreducible. When, on the other hand, the simple elements in question are related by inclusion, a regressed dichotomy is obtained. Thus when $b < a$, we obtain a regressed dichotomy, in the form $a = a + b$, and when $a < b$, in the form $a = ab$.

Complex elements are here reduced to simple ones, and there can be no question of a really full and developed architectonics. The circumstance that complex elements are here reduced to simple ones leads to the result that 1 and 0 give again complex elements, for if, for example,

$a < b$, this is equivalent not only that ab is reduced to $a (ab = a)$, and $a + b$ to $b (a + b = b)$, but also that 1 reduces to $a' + b$ $(1 = a' + b)$, and 0 to ab' $(ab' = 0)$, as we see in equation I (p.22).

If, however, inclusive relations are absent, i.e., if $b \nless a$, $b' \nless a$, $a \nless b$, and $a \nless b'$, then none of the eight additive ($a + b$, etc.) or multiplicative (ab, etc.) complex elements, can be reduced to one of their components or factors, and, at the same time, 1 and 0 cannot be reduced to any of the complex elements. The system characterized by $a \nless a'$ and $a' \nless a$ will then represent a completely full and developed hemidialectic logic, and then the simple elements $\underline{a}, \underline{a'}, \underline{b}, \underline{b'}$ will be really coordinate elements, constituting a system of complex elements different from them.

The considerations developed above may be recapitulated as follows. General, universal logic, or pan-logic, should be distinguished from specialized logic. In pan-logic, the question of the mutual relations of simple elements is left entirely open, and this applies both to the relations of antithetic elements (of the same denomination) and of simple elements of different denominations. On the other hand, in logics representing specifications of universal panlogic, these relations are clearly defined, and this specification of these relations (in the first place of relations between a positive element and its negation) is responsible for removing the indefiniteness characteristic of a pan-logic system. We have distinguished two fundamental specifications of general logic, viz., hemidialectic logic, in which an inclusive relation does not exist between the positive element and its negation (in the widest sense of the term), and specifically dialectic logic, in which such a relation exists, and may appear in three forms: either $a < a'$ (but $a' \nless a$), or $a' < a$ (but $a \nless a'$), or $a < a'$ and $a' < a$, i.e., $a = a'$. Each of these special logics is further differentiated by the relations existing between elements of different denominations ($\underline{a}, \underline{b}$). This differentiation leads, in the case of hemidialectic logic, amongst others to an architectonic type of logic, possessing the greatest degree of completeness and development, and having the greatest number of unequivalent elements (as has been shown, this number is 16 for two-dimensional logic; cf. Table II, p.39). At the other extreme we have a second limiting system of logic, which is reduced and contracted to a minimum, and in which all the elements are mutually equivalent. This is the dialectic system, in which $a = a'$, and $b = b'$. In this way we obtain the following specifications of general logic.

I. Hemidialectic logic ($a \nless a'$, $a' \nless a$, $b \nless b$, $b' \nless b$).

I^a. $b \nless a$, $b' \nless a$, $a \nless b$, $a \nless b'$.

(System I^a is the most developed, and possesses the maximum number

of unequivalent elements.)

I^b. Of the relations of inclusion, mentioned as absent in I^a, the various relations of this kind and their combinations enter into this group.

II. Dialectic logic [$a < a'$ (but $a' \nless a$), or $a' < a$ (but $a \nless a'$), or $a = a'$, and similarly for b].

II^a. $a < a'$ (but $a' \nless a$), or $a' < a$ (but $a \nless a'$), and similarly for b, with different relations of inclusion or non-inclusion between a and b.

II^b. $a = a'$ and $b = b'$.

(System II^b is the most highly reduced, all its elements being mutually equivalent.)[1]

The ramifications of pan-logic may be more briefly characterized by employing the specifications of unity (or zero). The general definition of unity is $a < 1$, i.e., that any element is contained in 1. However, the question remains open as to whether any element does not in turn itself contain unity, and whether we have not then to deal with an equivalent of 1. Numerous possibilities, specifying the particular systems of logic, may here arise. On the basis of our previous considerations, the specifications of logic may also be presented as follows.

I. Hemidialectic logic. Unity (1) is not equivalent to the simple elements constituting it.

I^a. Unity (1) is not equivalent to any element other than itself.

I^b. Unity (1) is equivalent to an element which is not a simple element constituting it.

II. Dialectic logic. Unity (1) is equivalent to simple elements constituting it.

II^a. Unity (1) is equivalent to one of two antithetic simple elements constituting it.

II^b. Unity (1) is equivalent to both of the antithetic simple elements constituting it; it hence follows that unity is equivalent to each of the elements in general.

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[1] The equivalences remaining, up to the equivalence $1 = 0$ inclusive, follow from the above. It follows from them, first of all, that $a = b$. Since $1 = a + a' = b + b'$, then, substituting a for a' and b for b', we receive $1 = a = b$. Similarly, we receive $0 = a = b$, and so $0 = 1$.

C H A P T E R IX.

Spatial Forms of the Specifications of Pan-Logic

We have so far examined the specifications of pan-logic from the point of view of pure logic; we shall now supplement our findings by examining the geometrical aspect of these specifications. The query at once arises: Does general logic - structural pan-logic - find express in any spatial form? (It will be remembered that pan-logic splits up into dialectical and hemidialectical logic,with their further sub-divisions.) Just as we cannot trace out the shape of a triangle in general (although we can draw right-angled,acute and obtuse-angled),so it is not possible to give a depiction of general logic. We can only represent it as the result of a problem which is insoluble spatially,just as we represent (but do not imagine) a triangle in general,by drawing a triangle (let us say a right-angled one) and in our mind depriving it of any qualitatively determined angles - keeping these indeterminate. Similarly,the general architectonics of the logical world can be rep(?)sented (not imagined) by maintaining contact with the spatial image in such manner that we set up as the basis the only so far known depiction of already specified architectonics,and deprive it of specific determinations,leaving the respective qualitative properties in an indeterminate state. Up to this point,we have dealt only with hemidialectical logic in its the most developed form which was spatially depicted for two and for three elements. It corresponds to type I^a of logic,which if we restrict ourselves to two elements,determines the following relations: $a \nless a'$, $a' < a$, $b < b'$, $b' \nless b$,and $b \nless a$, $b' \nless a$, $a \nless b$, $a \nless b'$ (cf.p. 78). Utilizing this picture of specified logic as the basis (see Fig.3), we can rise,without losing contact with the spatial sphere,to the representation (but not to the imagination) of two-dimensional logic in general,changing the above fixed relations between the simple elements $\underline{a}$,$\underline{a'}$, $\underline{b}$,$\underline{b'}$ for indeterminate relations. We shall be able to depict every specification of this general logic by appropriately changing the relations of our basic picture for other (but no less determinate) relations. In view of this,we must first of all realize precisely and clearly in what ways we can spatially depict the relations characterizing that specification of general logic,which we shall take as our point of issue - the fullest and the most developed logic (Type I^a,cf. Fig.3).

The relations in logic of Type I^a between the antithetic elements $a - a'$ and $b - b'$, viz., $a \nless a'$, $a' \nless a$, $b \nless b'$, $b' \nless a$, and $b \nless a$, $b' \nless a$, $a \nless b$, and $a \nless b'$ are expressed spatially by the parallelism of straight lines depicting the pairs $a - a'$ and $b - b'$, as also by the perpendicularity of the straight lines a and a' in respect of b and b', and dually by the position on the axis of co-ordinates of the points dual to them, as also by the equi-distance of the points a and a' with respect to b and b' (position of the points a and a' on the straight line which is the axis of symmetry for the points b and b'). The above spatial characterization of Type I^a logic can be supplanted by another, its equivalent one, received by the analytical characterization of this logic not by examining the relations of inclusion between its simple elements but the connexions between these elements (equivalents of these relations) in accordance with the principle:

$$a < b = (a + b = b) = (ab = a) = (ab' = 0) = (a' + b = 1).$$

It then appears, as we already know for that matter, that in Type I^a logic the complex elements can be neither reduced to one of its components or factors, nor to unity, nor to zero; moreover, unity and zero likewise cannot be reduced to simple elements. For instance, if we take element ab, it cannot be in this case reduced to a (since $a \nless b$), to b (since $b \nless a$), or to 0 (since $a \nless b'$); with the other elements the same holds good. The matter can now be examined spatially. Fig.3 throws light on this point: the straight line ab does not coincide with the straight line a, with the straight line b, or with the axis 0 (aa' and bb'). Similarly, the point $a + b$ does not coincide with the point a, the point b, or with the point 1 ($a + a'$ and $b + b'$); similarly with the other elements. We can therefore spatially characterize logic of this type by the fact that none of its elements coincide with any other - every one really occupies its own, separate position; this form of characterization fully replaces the spatial representation of Type I^a logic by means of the perpendicularity of the parallel straight lines a and a' with respect to the parallel straight lines b and b', as also by the equi-distances of the points a and a' on the axis $0_{aa'}$ with respect to the points b and b' on the axis $0_{bb'}$.

If we now, taking the spatial image of this most developed hemidialectical logic (Type I^a) as our point of issue, shall endeavour to represent another specification of this hemidialectical logic, two methods of doing this appear: one is partial and does not in reality change our basic spatial picture - it only indicates that it is now not such as it appears to us in the image; the second one is more radical, since for

every specification it yields a new picture, different to the basic one and received by means of changes effected with respect to the basic picture. In just the same way, as taking the acute-angled triangle ABC as our basis, we can utilize it to represent a right-angled triangle in two ways: once, by merely stating that, the angle, e.g. A, is not now acute but a right-angle; and once by actually changing the acute-angled triangle ABC into a triangle ABC which has a right-angle A - by transforming one shape into the other.

Thus, should we desire, on the basis of our image of fully developed logic, to represent one of the possible specifications of Type I^b (e.g., that in which $a < b$, but $b \nless a$, $b' \nless a$ and $a \nless b'$), then, abandoning the actual spatial depiction of this architectonical system, we shall be able best to effect its quasi-spatial representation by introducing algebraic (and not geometrical) equivalences which would characterize such system. These equivalences are yielded by the equation:

$$a < b = (a + b = b) = (ab = a) = (ab' = 0) = (a' + b = 1) = b' < a'^{1)} = \\ = (a' + b' = a') = (a'b' = b').$$

The quasi-depiction of the logic in question can therefore be presented as in Diagram 3, in which the following algebraic change can be effected: instead of $a + b$, we shall have there $a + b = b$ (which indicates that the point $a + b$ is not - as can be seen from the image - separate from point b, but coincides with it), and instead of ab we have $(ab = a)$, and instead of ab' we have $(ab' = 0)$ - which indicates that the straight line ab' coincides with the axis of co-ordinates, instead of $a' + b$ we have $(a' + b = 1)$ - which indicates that the point $a' + b$ coincides with the point at infinity of the axis of co-ordinates; finally instead of $a' + b'$ we have $(a' + b' = a')$ and instead of $a'b$ we have $(a'b = b')$.

In such wise, by utilizing the image of the most developed logic, we are able to represent to ourselves all the specifications of pan-logic; more even, this pan-logical system itself, without loss of contact with the spatial shape which binds all these systems of logic together. But these representations will be merely based on an intuitional foundation; we have here quasi-images - and not on real ones. As such they cannot suffice for our needs. For they include the clash between analytical and intuitional data, between what we have before our eyes and what we

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1) The equivalence $a < b = b' < a$ represents in this case the principle of contraposition: its validity follows from the fact that $a < b$, being equivalent to $ab' = 0$ (or $a' + b = 1$), hence it is equivalent to $b' < a'$.

should have. In our topologic, however, we desire to give an adequate spatial depiction of the relations of the logical world. We must therefore strive to give the spatial depictions (in the strict sense of the term) of all the specifications of architectonical logic; we must introduce into our basic diagram not only analytico-expressional changes but also spatial ones. We therefore turn to this matter.

To introduce spatial changes into the basic image of the logical plane, is tantamount to changing certain of its spatial qualities; to change certain points or directions is tantamount to shifting them. In such wise, by introducing movement into the image of the logical plane, we secure modifications which depict types of logic different to the fundamental one. We give below depictions of hemidialectical logic of Type I^b (Fig.12), and then of dialectical logic of Types II^a (Fig.13) and II^b (Fig.14).

Of the possible specifications of Type I^b we select that whose representation, attained without spatial modifications of the basic image of the logical plane has just been examined in detail. We shall now endeavour to secure the image proper of that logic which differs from the fundamental (i.e., the most developed) one, only in that it includes the relation $a < b$. This causes $a = ab$, i.e. that the direction a coincides with the direction ab. In order to attain this, we revolve (see Fig.3) the straight line a (extending beyond the points $a + b$ and $a + b'$) around the point a until it coincides with the straight line ab; this in turn evokes a number of effects. Namely: the point $a + b$, lying on the straight line a will find itself on the straight line $O_{bb'}$, in its upper part; that is to say, it will occupy the position of the point b (all the points of the axis $O_{bb'}$, in its upper part are categorially points b (see p.29). The coincidence $b = a + b$ is received in this way and it can then be seen that the straight line a actually only now passes through the point b (i.e. $a < b$). At the same time, when the straight line a assumes the direction ab, the point $1_{a+a'}$ (lying at infinity on the straight line a) takes up the position of the point $a' + b$ so that we receive $1 = a' + b$, lying on the intersection of the straight line ab with the straight line a' (see p.29). Moreover, too, as a result of this movement of the straight line a, the straight line aa' perpendicular to a, will assume the position of the straight line ab', so that we receive: $O_{aa'} = O = ab'$. In view of this, however, that the quality O is represented spatially not only by the axis $O_{aa'}$, but also by the axis $O_{bb'}$ (two straight lines not coinciding with each other but none the less equivalents), we have $ab' = O_{aa'} = O_{bb'}$. Effecting these shifts of the axis $O_{bb'}$ to the direction ab' by means of

a revolution of this axis around the point b',we again receive a number of equivalent shifts. Namely: the straight line b',perpendicular to $O_{bb'}$, will move to a'b' (b' = a'b'),its point a' + b' will occupy the position of point a' (a' + b' = a'),and then we secure intuitive conviction that the straight line b' passes through the point a' (i.e.,b' $<$ a'). In such manner we have spatially realized (not only by a process of thought or verbally) all the modifications which distinguish the logic of Type I^b from the fundamental Type I^a,and as an outcome we receive the image of logic of Type I^b shown in Fig.12.

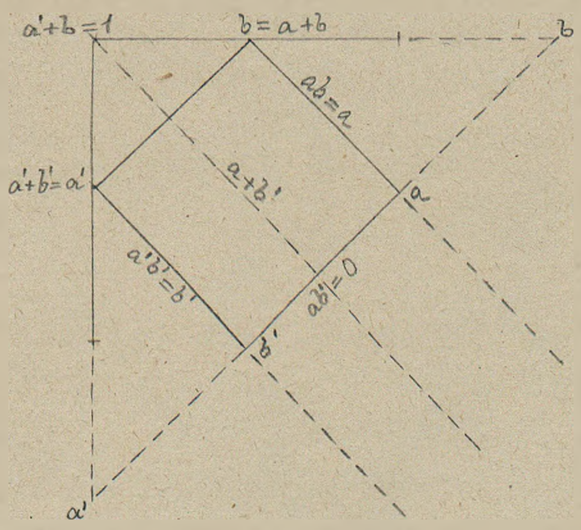

Fig. 12.

We now pass to dialectical logic Type II^a; of the two fundamental possible sub-types we shall select the one in which a $<$ a' and b $<$ b',and of the further specifications now possible we shall select the case in which a $<$ b',and hence b $<$ a'.[1] The relation a $<$ a' signifies spatially that the straight line a passes through the point a'. In order to introduce this fundamental modification into our basic depiction of the logical plane (Fig.3) we must revolve the straight line a by 90^o around the point a; it then assumes the position of the axis $O_{aa'}$ ($O_{aa'}$ = aa' = a) and will simultaneously pass through point a'. Simultaneously,the point at infinity on the straight line a ($1_{a+a'}$) - the point of intersection of the straight line a with the straight line a' (i.e.,a + a') - as a

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[1] This will be a logic,corresponding to the real relations existing,e.g. in the field of genetics when we examine the parents a + b and a' + b', of which the first element is a combination of gametes representing the same positive gene (a,b),and the second is a combination of gametes which represent the polar gene (a',b'),when the gene a' and b' is dominant (stronger,qualitatively greater),and a and b,is recessive (weaker, qualitatively smaller). We then have a $<$ a',b $<$ b',and a $<$ b' and b $<$ a'.

result of the above revolution of the straight line a will be located at point a' ($1_{a+a'} = a + a' = a'$), whilst the point a + b of this straight line occupies a position at the middle of the system of co-ordinates ($0 = 0_{aa'} + 0_{bb'} = a + b$). A modification of $b < b'$ can be similarly effected by revolving the straight line b through 90° around the point b, so that it passes through the point b'. We then receive: the axis $0_{bb'} = bb' = b$, and $1_{b+b'} = b + b' = b'$ [and $0 = 0_{aa'} + 0_{bb'} = a + b$]. We now have only to express spatially the modification $a < b'$ and $b' < a'$. With this in view we revolve the straight line a through another 45° around the point a, so that it coincides with the straight line ab' ($ab' = a$) and then a passes through the point b ($a < b'$), yielding at the intersection with the straight line b' the point b' (i.e. $a + b' = b'$) in accordance with the algebraic relations: $a < b' = (a + b' = b') = (ab' = a)$. Similarly, the straight line b is now revolved by another 45° around the point b so that it coincides with the straight line a'b ($a'b = b$), passing through the point a' ($b < a'$) and yielding at the intersection with the straight line a' the point a' ($b + a' = a'$). In such wise, we effect spatially all the modifications which distinguish one of the logics of Type II[a] from our basic logic, and as a result receive the following depiction of that logic, in which $a < a'$ and $b < b'$, as also $a < b'$ and hence $b < a'$ (Fig.13).

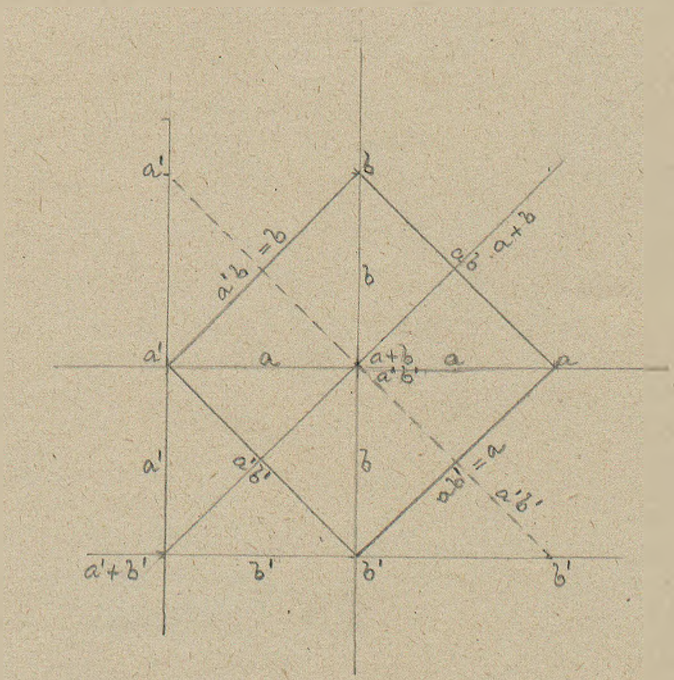

Fig. 13.

If we now compare Fig.13 with the basic Fig.3, we at once note a series of fundamental differences between them. Whilst in Fig.3 absolute symmetry rules, the picture in this case is quite different. First of all, there is no corresponding element to the straight line a', in the shape

of a parallel straight line a in relation to it symmetrically lying on the other side of the axis of co-ordinates; the same can be noted in the case of the straight line b. Further, the symmetry in the system of points on the straight lines also disappears; we can see in Fig.3, on every straight line, a central point which has on either side different symmetrically placed points, whilst on the straight lines in Fig.13 the central point divides them into two heterogeneous parts. Thus, for instance, the straight line a' above point a' has no other points apart from the point a', whilst on its lower part we see the point a' + b'; a similar asymmetry is noted on the straight line b' between its right and its left half. This asymmetry is still more in evidence on the slanting straight line a + b, passing through the origin of the axis of co-ordinates a + b; all the points on this straight line lying on one side of this point, viz., in the first quarter, are a + b, whilst on the other side of the point we have on the straight line a + b the point a' + b' and the point a'b' (at the intersection of the straight lines a + b and a'b'). These and similar asymmetries, which clearly distinguish the basic topological architectonics in Fig.3 from the dialectical topological architectonics of Fig.13, are the outcome of the non-coordination existing in the latter between the element a and a', b and a'. The element a' is stronger than a, and predominates over it as also over b; this is why the equilibrium and symmetry typical of our basic topologic vanishes. It is this predominance of the negative over the positive elements which determines the asymmetrical configuration of the point on the straight line a + b with its middle point also designated as a + b; in view of the fact that $a + b > a, a + b > b$, the straight line a + b cannot pass through the points a and b and hence here can be no points a and b in the positive part of this straight line; on the contrary even, the elements a and b in the shape of straight lines pass, as we see in the diagram, through the ~~pxixtxa~~ point a + b. On the other hand, since $a + b < a'b' < a' < a' + b'$, the ~~strai~~ straight line a + b in its negative part (in the fourth quarter) indicates the points a'b' and a' + b'.

One remark more must be made in this connexion. As can be seen from Diagram 13, no straight line parallel to a' passes through the point a; the straight line a, parallel to a' on Diagram 3, is in this case perpendicular to a'. Taking the point b and the straight line b', the same is evident. A plane, upon which no parallel line can be traced through a gi-
√point to a gi- ven√straight line, is a non-Euclidean, namely, a Rieman one. Categorial
ven straight lines - at least some of them - do not in this case extend into infinity; they have no limitary points which would represent infinite

qualities (1). The operations and relations between finite elements do not lead here beyond their sphere; it is a domain which has no boundaries behind it to which it might proceed; its is closed in itself and by itself; it is autarkic and finite in the ancient sense of the term.

Finally, we shall examine spatially dialectical logic (II^b), in which not only $a < a'$ but also $a' < a$ or $a = a'$ (similarly $b = b'$). In this case it is not necessary to have to do with Fig.3, but directly to base our examination on Fig.13, and to continue the reduction of its elements and their involution. Thus, the straight line a' is also now revolved around the point a' through 90°, so that is coincides with the axis a and passes through the point a; we proceed similarly with the straight line b'; all the vestiges of the outer square will then vanish and we receive as a result only the inner square having the diagonals (axes) $a = a'$ and $b = b'$ (Fig.14).

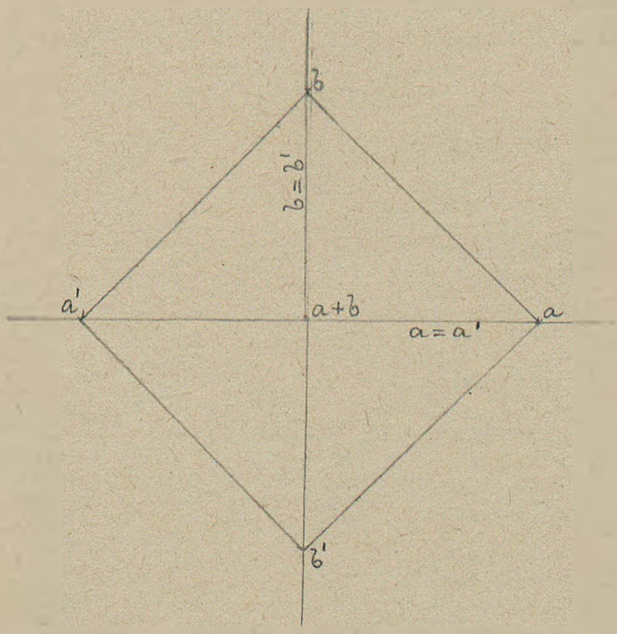

Fig. 14.

An extreme case of the logic in which $a = a'$ and $b = b'$ is received when the point a will not only be equivalent to the point a' but also identify itself with and coincide in it; similarly point b with point b'. The outer square will then also be involuted since the axis aa' will be condensed to its central point, to which also will be drawn the axis bb'. In such wise, all the elements of the logical plane are reduced to a single point - to the point of origin of the co-ordinates.

x x

x

It is quite natural that we have in our considerations hithertofore struck on traces of non-Euclidean logic (see pp.75 and 86). In very fact,since logic presents itself with such far-reaching coincidence with geometry (projective),then it is quite natural that logical specifications,which take into consideration the various possibilities arising between the logical elements depicted by parallel lines,will correspond to the geometrical specifications which take account of the various possibilities in connexion with such lines. As our geometrical logic is a categorial one,the positive element a (or b) is represented only once,by one point and by one straight line - similarly with the negative element. For this reason,the parallel lines are represented here not by two positive elements (or by two negative ones),but by two polar elements - one positive and the other negative (a,a' ; b,b').We shall now endeavour to enter a little more closely into the nature of that analogy evident in the existence of various systems,differing from each other because of the existence of parallel elements in the domains of logic and geometry.

If we comprehend space and its elements as something absolutely divorced from any concrete and dynamic substrate - as pure space - in which there are no inter-elemental influences,in which nothing happens, and in which everything intuitionally possible is hence also actual - the comprehension of the possibility of non-Euclidean geometry will in such case encounter insurmountable obstacles. We shall never in such event understand why a straight line - one having an absolutely constant direction,so natural for our geometrical intuition - should be impossible and have to be superseded by a geodetical curved line. We would not understand this since in pure,non-dynamic space we shall find no reasons for this and we shall be inclined to treat such ideas as analytical concepts,deprived not only of concrete but even of spatial significance. The situation,however,assumes quite a different aspect when,following in the tracks of Einstein,we approach space to the concrete world,permeate its elements with matter and energy,and make it physical and dynamic space. We shall then readily understand why in some circumstances Euclidean straight lines are impossible in such a space; that they will be - as in the case of light-rays passing from the stars to us - curved by the gravitative influence of the sun's mass near which they pass. Hence,if we desire that our geometrical elements adequately correspond to real relations,we cannot base ourselves solely on pure geometrical

intuition and disregard gravitational processes, but must take into consideration yet other possibilities, given us a priori in the specifications of the pangeometrical formulae, possibilities which we must comprehend as the formulation of physical forces (primarily gravitational ones).

The domain of logic presents itself in just the same light as in the case of geometry. Fully divorced from any actual, concrete basis, in pure logic no forces can exist which act between its elements; there are no stronger or weaker elements and none which perturbate or are perturbed; nothing happens in the domain of a logic which is so comprehended. If we have two elements a and a', there is no logical reason why a should be stronger than a' ($a' < a$) or conversely; there is no logical reason to recognize in general that any forces, operations or dynamic relations exist in the domain of pure logic, and its elements, such, for instance, as a and a', can be taken only as being quite independent of each other, as being quite equi-ordinate, as not entering each other's paths, or, to use the language of topologic - as non-intersecting, i.e. parallel. But if we desire logic to answer to all the real possibilities - that it represent them all - it must absorb real forces acting in the world, and, having thus dynamically become animated, give these real forces expression in logical operations and relations. In such a manner we must comprehend formulae of mathematical logic which, for that matter, first gave a place of paramount importance to the concept of logical "operations". We must comprehend these formulae just as dynamically as we dynamically comprehend pangeometrical ones - in just the same way and for the same reasons.

We shall then understand that the elements a and a' need not necessarily be independent of each other; that one of them, e.g., a' can represent a really stronger element ($a < a'$), which exerts its influence on the weaker element a, introduces perturbations and deviations into it, just as with the results of gravitational operation in the domain of physical geometry. In the case of this extreme predominance of the element a' over a, i.e. $a \lessdot a'$ (cf. footnote on p.85), the vertical straight line a (representing the logical weaker element a) is, as has been shown, fully diverted from its direction parallel to the vertical direction of the line a' and, assuming the direction of the horizontal axis, is absorbed by the point a' (cf. Fig.13). We can now fully understand that in this case ($a < a'$) there can be no topological straight line a parallel to straight line a'. It is true, there are no reasons in pure topologic, i.e. in pure logic and pure geometry, which would make the existence of such

a straight line out of the question - but in the given case there are real dynamic reasons, reasons of dynamic topologic, expressed in the formula $a < a'$, which make the existence of such a straight line impossible. In short, we cannot trace the straight line a parallel to the straight line a' through the topological point a - in the event of the existence of an element a' stronger than its pole a ; in such wise, we state the existence here of a non-homogenous, i.e., non-Euclidean logical space, to which will correspond the non-Euclidean geometrical logic (non-Euclidean ~~geometrical~~ geometry of logic).

APPENDIX

The Logic of Dichotomy and the Three Pythagorean Means

Taking the three Pythagorean means - the arithmetical, harmonic, and geometrical means - an attempt is made to demonstrate how greatly logical structures permeate the arithmetical world. The logical nature of these three fundamental arithmetical formations will be revealed, as will also be certain hitherto unknown relations amonst them, by means of a due realization of those qualitative structures of which they are the expression.

We already know (see p.65) that the arithmetical mean depicts the logical product and the harmonic mean - the logical sum. We must, however, now clearly realize that this correspondence is not absolute, but is governed by certain definite conditions.

It cannot be applied, for instance, when we desire to arithmetize a logical relation which expresses the equivalence between a sum (or logical product) and one of the components of such sum (or product), say, the relation (which does not always hold true): $a = a + b$ or $a = ab$, since the harmonic (or arithmetical) mean of two different numbers can never be equal to one of them. If then we take any logical formula which expresses the equivalence of the simple element to a given function of the elements (of which the simple elements is a component), the arithmetical depiction in question of such formula will only then be possible when we exclude cases which cause the given formula to assume the type: $a = a + b$ or $b = a + b$ (or $a = ab$ or $b = ab$).

If, for example, we desire arithmetically to depict along the above lines the principle of dichotomy:

$$a = (a + b)(a + b'),$$

this can only be done provided that $b \nless a, b' \nless a, a \nless b$, and $a \nless b'$. This is so because this condition is equivalent to stating that $a + b' \neq 1$, $a + b \neq 1$, $a + b \neq b$, and $a + b' \neq b'$, and this assures the above principle against reduction to a formation which cannot be arithmetized by the method indicated.

The same conditions hold good in the arithmetization of the principle of dichotomy dual with respect to the preceding principle.

This complete independence of the elements a and b, necessary for the above arithmetization of the principles of dichotomy, and expressed:

$$b \nless a,\ b' \nless a,\ a \nless b, \text{ and } a \nless b'$$

can be presented in yet another form, viz., by introducing the concept of the logical complete neutrality of one element with respect to the other two, polar ones. In the given case, this will be the complete neutrality of the element a [or a'] with respect to b and b' (or of b [or b'] with respect to a and a'). Actually, the above four conditions also express that the element a [or a'] inclines neither towards the element b nor towards the element b' (the negation or reciprocal of b), but occupies a neutral position between them, that is to say, it is "a logical complete neutral mean" between them. Thus we define: a logical complete neutral mean a with respect to b and b' =

$$(b \nless a) + (b' \nless a) + (a \nless b) + (a \nless b').$$

If in the field of numbers we now succeed in recognizing this logical complete neutrality, if we succeed in explaining it in terms of arithmetic (just as with the concepts of logical sum and product), the arithmetical representation of the principles of dichotomy should encounter no obstacles in its way.

But the arithmetical mean corresponds to the logical product and the harmonic mean corresponds to the logical sum; hence only the geometrical mean remains, it would appear, for the depiction of the neutral element, i.e. the logical complete neutral mean. This for that matter, so natural supposition will be found to be valid in its arithmetical applications.

We can now proceed to represent arithmetically the principles of dichotomy, utilizing our glossary:

Logical product - arithmetical mean,

logical sum - harmonic mean,

od nowego wiersza

logical complete neutral mean - geometrical mean, bearing in mind, too, the conditions to be respected by the elements appearing in this connexion.

We secure certain, hitherto unknown, relations joining up the three arithmetical means which can be formulated in the following theorems, in which, instead of the element $\frac{1}{b}$ (corresponding to the logical element b') we have a more general element in the shape of c.[1]

T h e o r e m 1.[2]

Let a be the geometrical mean of b and c; then the arithmetical

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1) Just as in further theorems, instead of the element $\frac{1}{a}$ (corresponding to the logical element a'), a more general elements appears in the shape of d.

2) Corresponding to the logical theorem: Let a be the complete neutral mean of b and b'; then $a = (a + b)(a + b')$, where $a + b$ and $a + b'$ can be reduced neither to 1 nor to simple elements b, b' (conditions for the arithmetization of the principle of dichotomy).

mean of two harmonic means (1) that of a and b and (2) of a and c is equal to a (Theorem Iα), and conversely (Theorem Iβ).

P r o o f .

(Iα), H_{ab} = harmonic mean of a and b = $\frac{2ab}{a+b}$;

A_{ab} = arithmetical mean of a and b = $\frac{a+b}{2}$

$$A_{H_{ab}\cdot H_{ac}} = \frac{\frac{2ab}{a+b}+\frac{2ac}{a+c}}{2} = \frac{ab}{a+b}+\frac{ac}{a+c} =$$

$$= \frac{ab(a+c)+ac(a+b)}{(a+b)(a+c)} = \frac{a^2b+2abc+a^2c}{a^2+ab+ac+bc}$$

If $bc = a^2$, then

$$A_{H_{ab}\cdot H_{ac}} = \frac{a^2b+2a.a^2+a^2c}{a^2+ab+ac+a^2} = \frac{a^2(2a+b+c)}{a(2a+b+c)} = a.$$

(Iβ). $A_{H_{ab}\cdot H_{ac}} = \frac{a^2b+2abc+a^2c}{a^2+ab+ac+bc} = a$

$$a^2b + 2abc + a^2c = a^3 + a^2b + a^2c + abc$$
$$abc = a^3$$
$$bc = a^2$$

T h e o r e m II (dual to Theorem I).[1)]

Let a be the geometrical mean of b and c; then the harmonic mean of two arithmetical means (1) that of a and b, and (2) of a and c, is equal to a (Theorem IIα), and conversely (Theorem IIβ).

P r o o f.

(IIα).

$$H_{A_{ab}\cdot A_{ac}} = \frac{2\left(\frac{a+b}{2}\right)\left(\frac{a+c}{2}\right)}{\frac{a+b}{2}+\frac{a+c}{2}} = \frac{(a+b)(a+c)}{a+b+a+c}$$

$$= \frac{a^2+ab+ac+bc}{2a+b+c}$$

If $bc = a^2$, then

$$H_{A_{ab}\cdot A_{ac}} = \frac{a^2+ab+ac+a^2}{2a+b+c} = \frac{a(2a+b+c)}{2a+b+c} = a$$

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1) Corresponding to the logical theorem: Let a be the complete neutral mean of b and b'; then a = ab + ab', where ab and ab' cannot be reduced to 0 or to simple elements b, b'.

(IIβ). $$H_{A_{ab} \cdot A_{ac}} = \frac{a^2 + ab + ac + bc}{2a + b + c} = a$$

$$a^2 + ab + ac + bc = 2a^2 + ab + ac$$

$$bc = a^2$$

Conclusion from Theorems I and II.

Let $\underline{a}$ be the geometrical mean of $\underline{b}$ and $\underline{c}$: then the arithmetical mean of the harmonic means of $\underline{a}$ and $\underline{b}$, and of $\underline{a}$ and $\underline{c}$ is equal to the harmonic mean of the arithmetical means of the above elements, and conversely.

We can determine further, more complex relations between the geometrical, arithmetical and harmonic means.

Theorem III.[1)]

Let $\underline{c}$ be the geometrical mean of $\underline{a}$ and $\underline{d}$, and $\underline{d}$ that of $\underline{c}$ and $\underline{b}$; then the arithmetical mean of $\underline{c}$ and $\underline{d}$ is equal to the arithmetical mean of the four harmonic means: of the harmonic mean of $\underline{a}$ and $\underline{c}$, of the harmonic mean of $\underline{b}$ and $\underline{d}$, and of the twice-taken harmonic mean of $\underline{c}$ and $\underline{d}$.

Proof.

If $c^2 = ad$, $c = A_{H_{ac} \cdot H_{cd}} = \frac{H_{ac} + H_{cd}}{2}$ (Theorem I)

If $d^2 = cb$, $d = A_{H_{bd} \cdot H_{cd}} = \frac{H_{bd} + H_{cd}}{2}$ (Theorem I)

Hence

$$A_{cd} = \frac{H_{ac} + H_{bd} + H_{cd} + H_{cd}}{4} = \frac{H_{ac} + H_{bd} + 2H_{cd}}{4}$$

Theorem IV (Dual to Theorem III)[2)]

Let $\underline{c}$ be the geometrical mean of $\underline{a}$ and $\underline{d}$, and $\underline{d}$ that of $\underline{c}$ and $\underline{b}$; then the harmonic mean of $\underline{c}$ and $\underline{d}$ is equal to the harmonic mean of the

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1) Corresponding to the logical theorems:

$$c = (c + a)(c + a') \text{ and } d = (d + b)(d + b')$$

whence

$cd = (c + a)(d + b)(c + a')(d + b')$, respecting the conditions for the arithmetization with the generalizations:

$$\frac{1}{a} = d \quad \text{and} \quad \frac{1}{b} = c.$$

2) Corresponding to the logical theorems:

$$c = ca + ca', \quad \text{and } d = db + db'$$

whence

$c + d = ca + db + ca' + db'$, respecting the conditions for their arithmetization with the generalization:

$$\frac{1}{a} = d \quad \text{and} \quad \frac{1}{b} = c.$$

four arithmetical means:[1] of the arithmetical mean of a and c, of the arithmetical mean of b and d, and of the twice-taken arithmetical mean of c and d.

P r o o f.

If $c^2 = ad$, then $c = H_{A_{ac} \cdot A_{cd}} = \dfrac{2A_{ac} \cdot A_{cd}}{A_{ac} + A_{cd}}$ (Theorem II)

If $d^2 = cb$, then $d = H_{A_{bd} \cdot A_{cd}} = \dfrac{2A_{bd} \cdot A_{cd}}{A_{bd} + A_{cd}}$ (Theorem II)

Hence,

$$H_{cd} = \frac{2\left(\dfrac{2A_{ac} \cdot A_{cd}}{A_{ac} + A_{cd}}\right) \left(\dfrac{2A_{bd} \cdot A_{cd}}{A_{bd} + A_{cd}}\right)}{\dfrac{2A_{ac} \cdot A_{cd}}{A_{ac} + A_{cd}} + \dfrac{2A_{bd} \cdot A_{cd}}{A_{bd} + A_{cd}}} =$$

$$= \frac{2(2A_{ac} \cdot A_{cd})\ (2A_{bd} \cdot A_{cd})}{2A_{ac} \cdot A_{cd}(A_{bd} + A_{cd}) + 2A_{bd} \cdot A_{cd}(A_{ac} + A_{cd})}$$

$$= \frac{4 \cdot A_{ac} A_{bd} \cdot A_{cd} \cdot A_{cd}}{A_{ac} \cdot A_{bd} \cdot A_{cd} + A_{ac} \cdot A_{cd} \cdot A_{cd} + A_{ac} \cdot A_{bd} \cdot A_{cd} + A_{bd} \cdot A_{cd} \cdot A_{cd}}$$

$$\left[= \frac{4 \cdot A_{ac} \cdot A_{bd} \cdot A_{cd}}{2A_{ac} \cdot A_{bd} + A_{ac} \cdot A_{cd} + A_{bd} A_{cd}} \right]$$

We can now make another step forward by basing ourselves on the following easily demonstrable logical proposition:

"If b' is a complete neutral mean of a and a', and a' is a complete neutral mean of b' and b[2], then a'b' is a complete neutral mean of ab' and a'b, and a' + b' is that of a + b' and a' + b."

If this is so, then the arithmetical theorem must also be true which we receive as the representation of the above logical theorem by the substitution of the geometrical mean for the neutral mean, of the harmonic means for logical sums, of arithmetical means for logical products, and generalizing as above $\frac{1}{a}$ = c and $\frac{1}{b}$ = d. Thus, in actual fact, we can demonstrate the following arithmetical theorem.

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1) Based on the definition of the harmonic mean of the given numbers as the reciprocal of the arithmetical mean, taken for the reciprocal of such numbers; we thus receive the formula for the harmonic mean of four elements, viz.:

$$H_{abcd} = \frac{4abcd}{abc + abd + acd + bcd}$$

2) Omitting this second condition, we likewise receive a valid proposition, since both these conditions are logically equivalent, as equivalent to the set: $b \not< a$, $b' \not< a$, $a \not< b$, $a \not< b'$. Omotting it, however, we deprive ourselves of the possibility of attaining the following generalized arithmetical propositions.

T h e o r e m V.

Let c be the geometrical mean of a and d, and d that of c and b; then the arithmetical mean of c and d is the geometrical mean of the arithmetical means of a and c, and of b and d.

P r o o f.

$$A_{cd} = \frac{c+d}{2}\ ;\ A_{ac} = \frac{a+c}{2}\ ;\ A_{bd} = \frac{b+d}{2}$$

$$[A_{cd}]^2 = \frac{(c+d)^2}{4}$$

$$A_{ac} \cdot A_{bd} = \frac{(a+c)(b+d)}{4} = \frac{ab+cb+ad+cd}{4}$$

If $c^2 = ad$, $d^2 = cb$; then, $ab = cd$; therefore

$$A_{ac} \cdot A_{bd} = \frac{cd+d^2+c^2+cd}{4} = \frac{(c+d)^2}{4} = [A_{cd}]^2.$$

T h e o r e m VI. (Dual to Theorem V).

Let c be the geometrical mean of a and d, and d that of c and b; then the harmonic mean of c and d is the geometrical mean of the harmonic means of a and c, and of b and d.

P r o o f.

$$H_{cd} = \frac{2cd}{c+d}\ ;\ H_{ac} = \frac{2ac}{a+c}\ ;\ H_{bd} = \frac{2bd}{b+d}$$

$$[H_{cd}]^2 = \frac{4c^2d^2}{(c+d)^2}.$$

$$H_{ac} \cdot H_{bd} = \frac{4acbd}{(a+c)(b+d)} = \frac{4ab \cdot cd}{ab+cb+ad+cd}$$

If $c^2 = ad$, $d^2 = cb$, then $ab = cd$;

therefore,

$$H_{ac} \cdot H_{bd} = \frac{4cd \cdot cd}{cd+d^2+c^2+cd} = \frac{4c^2d^2}{(c+d)^2} = [H_{cd}]^2.$$

Applying Theorems VI and V to Theorems III and IV, the following theorems are yielded.

T h e o r e m VII.

Let c be the geometrical mean of a and d, and d that of c and b; then the square root of the arithmetical mean c and d is equal to the arithmetical mean of the square roots of the harmonic means a and c, and b and d.

P r o o f.

If $c^2 = ad$, $d^2 = cb$, then

$$A_{cd} = \frac{H_{ac} + H_{bd} + 2H_{cd}}{4} \qquad \text{(Theorem III) and}$$

$$H_{cd} = \sqrt{H_{ac} \cdot H_{bd}} \qquad \text{(Theorem VI)}$$

Therefore,

$$A_{cd} = \frac{H_{ac} + H_{bd} + 2\sqrt{H_{ac} \cdot H_{bd}}}{4} = \left(\frac{\sqrt{H_{ac}} + \sqrt{H_{bd}}}{2}\right)^2$$

$$\sqrt{A_{cd}} = \frac{\sqrt{H_{ac}} + \sqrt{H_{bd}}}{2}.$$

T h e o r e m VIII (Dual to Theorem VII)

Let c be the geometrical mean of a and d, and d that of c and b; then the square root of the harmonic mean c and d is equal to the harmonic mean of the square roots of the arithmetical means a and c, and b and d.

P r o o f.

If $c^2 = ad$, $d^2 = cb$, then

$$H_{cd} = \frac{4A_{ac} \cdot A_{bd} \cdot A_{cd}}{2A_{ac} \cdot A_{bd} + A_{ac}A_{cd} + A_{bd}A_{cd}} \qquad \text{(Theorem IV) and}$$

$$A_{cd} = \sqrt{A_{ac} \cdot A_{bd}} \qquad \text{(Theorem V)}$$

Therefore,

$$H_{cd} = \frac{4A_{ac} \cdot A_{bd}\sqrt{A_{ac} \cdot A_{bd}}}{2A_{ac} \cdot A_{bd} + A_{ac}\sqrt{A_{ac}A_{bd}} + A_{bd}\sqrt{A_{ac} \cdot A_{bd}}} =$$

$$= \frac{4A_{ac}A_{bd}\sqrt{A_{ac} \cdot A_{bd}}}{\sqrt{A_{ac} \cdot A_{bd}}\left(2\sqrt{A_{ac}A_{bd}} + A_{ac} + A_{bd}\right)} =$$

$$= \frac{\left(2\sqrt{A_{ac} \cdot A_{bd}}\right)^2}{\left(\sqrt{A_{ac}} + \sqrt{A_{bd}}\right)^2}$$

$$\sqrt{H_{cd}} = \frac{2\sqrt{A_{ac} \cdot A_{bd}}}{\sqrt{A_{ac}} + \sqrt{A_{bd}}}$$

C o n c l u s i o n f r o m T h e o r e m s VII a n d V.

Let c be the geometrical mean of a and d, and d that of c and b; then the arithmetical mean of the square roots of the harmonic means a and c, and b and d is the geometrical mean of the square roots of the arithmetical means of the same numbers.

C o n c l u s i o n f r o m T h e o r e m s VIII a n d VI (dual to the above).

Let c be the geometrical mean of a and d, and d that of c and b; then the harmonic mean of the square roots of the arithmetical means a and c, and b and d is the geometrical mean of the square roots of the harmonic means of the same number.

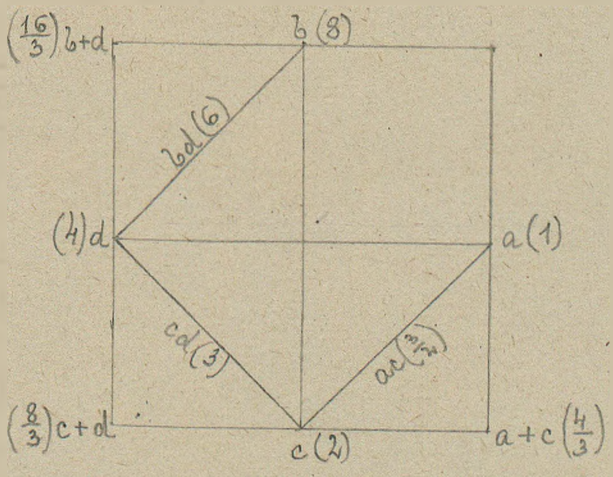

Fig. 15.

We can now represent the above theorems diagramatically (changing in the first two theorems, b - a - c for a - c - d); for instance, for the numbers 1,2,4,8 where 2 and 4 are two geometrical means between 1 and 8. We shall here have: a = 1, c = 2, d = 4 and b = 8.

In such wise we conclude our examination of the relation between the field of structural logic and arithmetics. We have gained the conviction that there is a most profound and remarkable correspondence between these fields, and that the exact analogy which binds together the world of concepts with the world of numbers makes it possible for us to discover hitherto unknown albeit fundamental arithmetical relations.